Droplet theory in low dimensions: Ising systems in zero field

Droplet theory in low dimensions: Ising systems in zero field

A D Bruce and D J Wallace

University of Edinburgh, Department of Physics, Mayfield Road, Edinburgh EH9 3JZ, Scotland, UK

Abstract. We develop a theory of the universal configurational physics underlying critical-point phenomena in the Ising universality class. The theory is formally justifiable in $d = 1 + \varepsilon$ dimensions and may be regarded as the natural continuation of the kink-based theory of one dimension, which it incorporates as a limiting case. In $d = 1 + \varepsilon$ the configurational building block is the droplet. The typical droplet is not spherical and the many-droplet assembly is not dilute: the implied problems are handled with renormalisation group methods. It is found that droplet shape fluctuation effects control the correlation length exponent ν, while the nesting of droplets within droplets controls the order parameter exponent β. The exponents ν and β thus effectively define, respectively, the fractal dimensions of the droplet surface and the droplet volume. The theory is used to determine and illuminate the critical behaviour of further quantities including the free energy, the susceptibility, the droplet number distribution and the distribution of the intra-droplet order.

1. Introduction

It is now widely recognised that the long-wavelength behaviour of a system undergoing a continuous phase transition is remarkably insensitive to its specific microscopic characteristics. The recognition of this, the universality phenomenon, is at the heart of contemporary theories of the critical region, in which it is to varying degrees vindicated and exploited (Kadanoff 1971, Wilson and Kogut 1974, Fisher 1974). At a formal level the phenomenon is defined by the assertion that the correlation functions of (arbitrary numbers of) ordering coordinates, separated by distances large compared with all microscopic lengths, have universal forms (forms which are the same for all members of a universality class) modulo two non-universal scales (Bervillier 1976). At a more physical level the phenomenon may be expressed in the statement that the spectrum of coarse-grained configurations is universal, modulo ordering-coordinate and spatial coordinate scales (Bruce 1981a).

The notion that the universality of the critical region resides, ultimately, in the universality of the underlying coarse-grained configurations is both challenging and potentially fruitful. The challenge it poses lies in the task of characterising the universal patterns and in identifying such links as may exist between specific configuration features and the values of the more familiar universal observables, typified by critical exponents. The idea is a potentially fruitful one in as much as it illuminates the need

for critical-point approximation schemes to be tailored to respect as closely as possible the (anticipated) coarse-grained configuration spectrum. Moreover, it engenders the helpful expectation that the physics will be tractable when the patterns are simple.

These general remarks represent an amalgam of the motivation for and lessons learned from the study reported in this paper, which amplifies an earlier brief communication (Bruce and Wallace 1981). We develop a configuration-based theory of the phase transition exhibited by members of the Ising universality class in low space dimensions. By 'Ising universality class' we mean systems characterised by a scalar order parameter and short-range interactions. By 'low' space dimensions we mean, formally, dimensions d for which $\varepsilon = d - 1$ may be regarded as a small parameter.

If one regards the theory merely as a computational scheme the latter constraint is a severe one: in particular, it seems unlikely that the theory can be persuaded to yield numerically reliable predictions for critical exponents, even for $d = 2$. It seems probable, however, that some of the more general lessons and insights which emerge from the theory do transcend this limitation: it is thus at a conceptual rather than narrowly computational level that the value of the theory should be assessed.

The utility of $\varepsilon = d - 1$ as an expansion parameter is immediately plausible from a configurational standpoint. In one dimension the coarse-grained configurational physics is, universally, that prescribed by a gas of kinks (Krumhansl and Schrieffer 1975, Zinn-Justin 1981, Bruce 1981b); the gas is dilute so that the spectrum of configurations, and the collective behaviour in which they are manifested, are easily determined. It transpires that the dominant role of the kink and the simplicity of the dilute kink gas have natural analogues in $d = 1 + \varepsilon$ dimensions. The analogue of the kink is the boundary of a *droplet* of one phase embedded in a background of the opposite phase. The analogue of the diluteness of the one-dimensional kink gas is the *diluteness of droplet boundaries*. As a result, an explicit analytically tractable droplet-based theory is realisable for $d = 1 + \varepsilon$.

The utility of a 'droplet' as an elementary configuration-building block in Ising systems has long been appreciated. The earliest references are cited by Fisher (1967); recent reviews have been given by Domb (1976) and Binder (1976). To date, the concept has proved most powerful in the context of sub-critical coexistence curve behaviour: droplet-based theories predict a (conceptually if not practically) significant essential singularity in thermodynamic potentials on the line of phase coexistence—the line $T < T_c$, $H = 0$ in the language of magnetism (Andreev 1964, Fisher 1967, Langer 1967, Günther *et al* 1980). This singularity is not readily captured in alternative theories, although extensive series expansion studies (Baker and Kim 1980, Enting and Baxter 1980) and Monte Carlo work (Jacucci *et al* 1983) have been shown to be consistent with the predictions of the most recent and most detailed droplet-based theory of this phenomenon (Lowe and Wallace 1980). The source of the success of the simple droplet picture in this context is not hard to identify: the predicted singularity may be traced to the contributions, to thermodynamic potentials, made by droplets (of the phase opposite to that favoured by the vanishingly small field H) whose size is sufficiently *large* that the bulk (field-controlled) droplet energy competes with the surface energy. Now droplets which are large—in particular, large on the scale of the correlation length ξ—are simple in two respects. Firstly, they are geometrically well-defined objects whose deviations from hyperspherical form are statistically negligible. This statement is vindicated in the course of the work described here. Secondly, large droplets clearly occur with low probability, thus rendering plausible the use of a dilute droplet gas approximation. This statement requires rather more elaborate

justification which emerges in the extension of the present work to the case of a finite ordering field. This extension will be described elsewhere; the present work is restricted to the coexistence curve.

Within the critical region itself the droplet concept has proved rather less powerful. In this regime, prior to the present work, it has provided little more than an illuminating scaling phenomenology of critical-point behaviour (see e.g. Binder 1976). The reason is clear: the configurational physics near the critical point does not have the simplifying features identified above. Specifically, to the extent that it is at all describable within a droplet framework, the critical behaviour must reflect the contributions of a *critically growing population* of droplets of a whole range of sizes including (indeed, pre-dominantly) droplets whose sizes are *small* compared with ξ. These observations identify two key problems. The need to incorporate the effects of small droplets raises the first problem: one may anticipate that small droplets are *not* generally hyper-spherical. Indeed, this recognition is implicit in the earliest droplet phenomenologies (Fisher 1967) which allow for the possibility that the (mean) surface area of droplets of a given volume scales with that volume in a non-trivial way. The second problem resides in the fact that the droplet gas will not, in any obvious sense, be 'dilute' near the critical point: excluded volume and droplet overlap effects are clearly inevitable. This recognition is embodied in a later variant of the phenomenological droplet picture (Kadanoff 1976a) which notes, in particular, the possibility of droplet 'nesting' ('droplets within droplets . . .').

The work presented here makes it clear that an adequate treatment of these two problems is an essential prerequisite of any droplet-based theory of the critical region: it transpires that (in $d = 1 + \varepsilon$) droplet shape fluctuations are instrumental in controlling the correlation length exponent ν while droplet nesting controls the order parameter exponent β. Each of these effects poses a problem of many length scales; each thus demands the use of renormalisation group (RG) methods. It turns out that (in low d) the two problems can be addressed separately, and they are most conveniently treated with rather different forms of renormalisation group. These remarks are reflected in the structure and flavour of this paper, which we now outline.

Sections 2, 3 and 4 are devoted to the properties of a single isolated droplet. The problems associated with droplet shape fluctuation effects are treated with the aid of a renormalisation scheme based upon dimensional regularisation and legitimised by a small ε approximation. This analysis may be viewed as an extension to the case of the 'almost spherical interface' of the theory of the 'almost planar interface' initiated by Wallace and Zia (1979). The analysis establishes the influence of shape fluctuations on the droplet geometry (surface area and volume) and population. The reader who wishes to skirt this, the most technical part of the present work, will find the key results summarised in § 4.1 where the single-droplet partition function (which, we shall ultimately see, determines the droplet population) is shown to have a universal scaling form controlled by a critical (correlation) length ξ.

In §§ 5, 6 and 7 we develop a *multi-droplet* description of the configurational physics of Ising systems. The key problems—droplet nesting and excluded volume—are handled with the aid of elementary RG arguments, which are realised in the form of simple differential equations. Again the analysis is justifiable for small ε—this time, by virtue of the diluteness of droplet boundaries (alluded to above) which turns out to be characteristic of low dimensions, even at the critical point. The results of the analysis include predictions for the order parameter (illuminating the configurational significance of β), the free energy and susceptibility (predictably confirming strong

scaling), the droplet number distributions (revealing an unexpected flaw in droplet phenomenologies) and the distribution of the magnetisation of a single droplet (illuminating the character of the universal configuration spectrum).

Finally, in § 8, we summarise the key points which emerge from our study, amongst which perhaps the most distinctive is the recognition that the two key exponents ν and β may be regarded, respectively, as controlling (or being controlled by) the *fractal* dimensions of the droplet surface and droplet volume, in low space dimensions.

2. The isolated droplet model

2.1. Preliminaries

Consider a fixed-length spin lattice Ising system. At zero temperature the system is occupied fully by one or other of the two ordered phases, to which we shall refer as 'black' and 'white'. Suppose that the boundary conditions are such that the favoured phase is white. Now suppose that the temperature is raised from zero. The equilibrium ensemble of configurations will then display clusters of black spins. Provided the temperature is low in comparison with T_c it is clear that these black clusters or 'droplets' will be effectively isolated from one another. This is the picture underlying the original phenomenological droplet theory (Fisher 1967). In this, the first part of the programme set out in § 1, we attempt to characterise, in a quantitatively predictive fashion, the configurational properties of such effectively isolated droplets. In the process we shall make three key assumptions which we now identify.

Firstly we shall assume that the configurational properties we seek may be extracted from a study of the *continuum limit* of the fixed-length-spin Ising model, realised as an appropriate field theory. The justification for the assumption lies in the expectation that the configurational properties of large droplets will display universal features which are independent of (for example) detailed lattice structure. The motivation for the assumption is that the corresponding field theory is technically more amenable (if conceptually more problematic) than its lattice counterpart.

Secondly, in formulating our model and identifying its properties we shall suppose that the statistically dominant droplets may be regarded as almost hyperspherical. This assumption proves correct (or, at least, self-consistent) 'near' dimension $d = 1$.

Thirdly we shall suppose that it is meaningful to pursue the study of isolated droplet properties into the critical region. The justification for this (apparently implausible) supposition will emerge in the course of the second part of our programme (beginning in § 5) where we shall argue that the dilute droplet picture, manifestly correct for $T \ll T_c$, actually remains appropriate (albeit in a subtly modified form) in the critical region, for low enough space dimension.

2.2. Droplet geometry

In the continuum limit of the lattice Ising model (or, equivalently, the deep well limit of the Landau–Ginzburg model) a 'droplet' is effectively a connected region of space with an intrinsically sharp closed boundary. In keeping with our preliminary remarks, we will characterise the shape of such a droplet through its deviation $f(\eta)$ from a hyperspherical reference droplet of radius R as in figure 1:
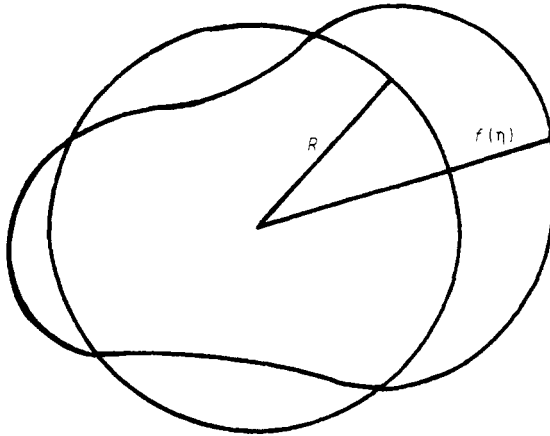
$$R(\eta) = R + f(\eta). \tag{2.1}$$

Figure 1. A nearly spherical droplet. The field $f(\eta)$ gives the deviation of the radial coordinate (in the direction η) from the radius R of a reference spherical droplet.

Here η specifies a direction from the centre of the reference droplet. The field f has a natural decomposition in the spherical harmonics $Y_{l\alpha}(\eta)$ of d dimensions:

$$f(\eta) = \sum_{l,\alpha} a_{l,\alpha} Y_{l\alpha}(\eta). \tag{2.2}$$

We now note a number of droplet properties which we shall require later (Günther *et al* 1980).

The volume contained within the droplet is given by

$$\mathcal{V} = \frac{1}{d} \int d\Omega \, (R + f(\eta))^d \tag{2.3}$$

where $d\Omega$ is the element of solid angle in d dimensions.

To obtain the surface area of the droplet it suffices to know the direction cosine $\boldsymbol{n} \cdot \boldsymbol{\eta} = \cos\theta$ between the unit normal $\boldsymbol{n}$ and the unit radial vector $\boldsymbol{\eta}$:

$$\mathcal{A} = \int d\Omega \, (R + f(\eta))^{d-1}/\cos\theta. \tag{2.4}$$

Suppose we have two neighbouring points $\boldsymbol{P} = R(\eta)\boldsymbol{\eta}$ and $P + \delta\boldsymbol{P}$ on the surface, related to one another by an infinitesimal rotation $\omega_{ij} \equiv -\omega_{ji}$ in the i–j plane. Then

$$\delta\boldsymbol{P} = \tfrac{1}{2}\omega_{ij}L_{ij}(R(\eta)\boldsymbol{\eta}),$$

where L_{ij} is the rotation generator in the i–j plane:

$$L_{ij} = x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i). \tag{2.5}$$

Using

$$\tfrac{1}{2}\omega_{ij}L_{ij}\eta_k = -\omega_{ki}\eta_i$$

the condition $\boldsymbol{n} \cdot \delta\boldsymbol{P} = \boldsymbol{0}$ for all antisymmetric ω_{ij} yields

$$(n_k\eta_i - \eta_k n_i)R(\eta) = L_{ki}R(\eta).$$

Squaring this equation and using the fact that n and η are unit vectors gives

$$(n \cdot \eta)^2 = R^2(\eta)[R^2(\eta) + \tfrac{1}{2}(L_{ij}R(\eta))^2]^{-1}. \tag{2.6}$$

Substituting into (2.4) and using (2.1) we obtain the explicit expression

$$\mathscr{A} = \int d\Omega \, (R + f(\eta))^{d-1}[1 + \tfrac{1}{2}(R + f(\eta))^{-2}(L_{ij}f(\eta))^2]^{1/2} \tag{2.7}$$

for the surface area of the droplet. The same expression can also be obtained directly from the induced metric on the surface.

Two droplets which are related by a rigid translation by vector x have of course the same value of $\mathscr{V}$ and $\mathscr{A}$. The fields f and f' which describe the two droplets are obviously different; for infinitesimal x and arbitrary f they are related by[†] (Günther *et al* 1980)

$$f'(\eta) = f(\eta) + x \cdot \eta + [R + f(\eta)]^{-1}x_i\eta_jL_{ij}f(\eta) + \mathrm{O}(x^2). \tag{2.8}$$

The expressions (2.3) and (2.7) are invariant under this transformation.

The properties of the spherical harmonics in d dimensions required for this paper can all be obtained from the fact that the symmetric traceless tensors of rank l form a basis for the functions $r^l Y_{l\alpha}$ for the given l. The number of such tensors gives the degeneracy

$$\nu_l(d) = \frac{\Gamma(l+d)}{\Gamma(d)\Gamma(l+1)} - \frac{\Gamma(l+d-2)}{\Gamma(d)\Gamma(l-1)} = \frac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(d-1)\Gamma(l+1)} \tag{2.9}$$

in terms of the Euler Γ function. The fact that the traceless tensors are eigenfunctions of the Laplacian operator

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + \frac{L^2}{2r^2}$$

gives directly the eigenvalues of $L^2 = L_{ij}L_{ij}$:

$$L^2 = -2l(l+d-2), \qquad l = 0, 1, 2, \ldots. \tag{2.10}$$

Two further remarks are appropriate here. Firstly we draw attention to the implicit assumption we have made that (statistically important) droplets can be described by (2.1) with a *single-valued* function f. The coherence of our subsequent perturbative calculations shows that this assumption is at least internally consistent (cf however, further remarks in § 8). Secondly, on a matter of terminology, we note that, although one cannot meaningfully speak of a droplet radius, one can (at least within our framework) associate with any droplet a unique length R which we shall term the droplet 'scale size'. It may be thought of as the radius of the underlying hyperspherical reference droplet (figure 1) from which the given droplet may be generated by assigning appropriate values to the amplitudes $a_{l\alpha}$ (equation (2.2)), which prescribe the field f, with the constraint $a_{0,0} = 0$.

[†] Equation (2.8) corrects a sign error in Günther *et al* (1980).

2.3. The single-droplet partition function

With the aid of (2.3) and (2.7) we may immediately formulate a model for the dimensionless configurational energy $\mathcal{H}$ of a single droplet relative to the zero-droplet ground state:

$$\mathcal{H} = \mathcal{H}(f) = \frac{1}{T_0} \int d\Omega \, (R + f(\eta))^{d-1} [1 + \tfrac{1}{2}(R + f(\eta))^{-2}(L_{ij}f)^2]^{1/2} + \frac{2H}{d} \int d\Omega \, (R + f(\eta))^d. \tag{2.11}$$

The first term represents the droplet surface energy: the parameter T_0^{-1} gives a measure of the surface tension. The second term represents the droplet volume energy: the ('magnetic') field H gives a measure of the distance from the coexistence curve. In the calculations reported here we set $H = 0$.

The configurational properties of the single-droplet ensemble are presecribed by the single-droplet partition function which we write as the functional integral

$$Z_1 = \int Df \, e^{-\mathcal{H}} \equiv \int \left(\prod_{l,\alpha} da_{l,\alpha} \right) e^{-\mathcal{H}}. \tag{2.12a, b}$$

The choice of measure implied by (2.12b) is important. In the course of our subsequent study of the many-droplet assembly we shall ascribe significance not only to the form of Z_1 but also to its overall scale. Specifically, we shall require that the normalisation of Z_1 is consistent with a zero-droplet (ground state) partition function that is precisely unity. We believe that the prescription (2.12b) fulfils this requirement within the framework of dimensional regularisation which we shall use (cf §§ 3.2, 3.3). The arguments by which we have been able to substantiate this belief are less complete than we would wish. They are set out in appendix 1.

3. Evaluation of the single-droplet partition function

In this section we set up perturbation theory for the single-droplet partition function defined by (2.11) and (2.12). The expansion parameter T_0 has dimension (length)$^{d-1}$; analytical control of the calculation is restricted to $d = 1 + \varepsilon$ dimensions. The necessary collective coordinate transformations are described in § 3.1, the perturbative calculation is given in § 3.2 and the renormalisation essential to the description of small droplets is performed in § 3.3.

3.1. Collective coordinate transformations

Conventional perturbation theory involves finding an extremum of $\mathcal{H}$ and expanding the field about its value at the extremum; anharmonic terms give systematic corrections to the Gaussian approximation. This approach cannot be followed directly in our case because there is no extremal droplet of non-zero radius; it is necessary to treat the scale size R of the droplet as a constrained collective coordinate. Further, since a rigid translation of the droplet does not change its energy, a factor proportional to the volume of space in which the droplet can be found must be extracted from the functional integral $\int Df$.

These aspects are reflected in the form of the expansion of the Hamiltonian (2.11) in a power series in f for a given fixed reference radius R:

$$\mathcal{H}(f) = T_0^{-1} \int d\Omega \, (R+f)^\varepsilon [1 + \tfrac{1}{2}(R+f)^{-2}(L_{ij}f)^2]^{1/2}$$

$$= (R^\varepsilon/T_0) \int d\Omega \, \{1 + \varepsilon R^{-1}f + \tfrac{1}{2}R^{-2}f[-\tfrac{1}{2}L^2 + \varepsilon(\varepsilon-1)]f + \mathrm{O}(f^3)\}. \tag{3.1}$$

Here we have written $\varepsilon = d - 1$ and have integrated by parts on L_{ij}. The first term in (3.1) gives the surface area of a sphere in d dimensions:

$$a_0 = R^\varepsilon S_d = 2\pi^{d/2} R^\varepsilon / \Gamma(d/2). \tag{3.2}$$

There is a term linear in f in (3.1) because $f = 0$ is not an extremum: $\int d\Omega \, f$ picks out only the $l = 0$ component of f in the decomposition (2.2). Using the eigenvalues (2.10) of L^2 we see that the quadratic term has a coefficient proportional to

$$-\tfrac{1}{2}L^2 + \varepsilon(\varepsilon-1) = l(l+\varepsilon-1) + \varepsilon(\varepsilon-1) \tag{3.3}$$

for a given l. For $l = 0$, (3.3) is $\mathrm{O}(\varepsilon)$ and for $l = 1$, it is $\mathrm{O}(\varepsilon^2)$; for $l \geq 2$ all eigenvalues are positive and $\mathrm{O}(1)$ or larger. Thus the modes with $l \geq 2$ can be handled by perturbation theory but the $l = 0$ and $l = 1$ modes cannot, because they would give factors of $\mathrm{O}(\varepsilon^{-1})$ and $\mathrm{O}(\varepsilon^{-2})$ in internal propagators which would invalidate a small-ε perturbation expansion.

To eliminate the $l = 0$ mode we note that the integration on a_{00} may be replaced by an integration over the droplet scale-size R. Specifically, since $Y_{00} = (S_d)^{-1/2}$ by normalisation, the $l = 0$ component of $R + f(\eta)$ is $R + a_{00}(S_d)^{-1/2}$. Thus we may make the replacement

$$\int da_{00} = (S_d)^{1/2} \int dR \tag{3.4}$$

so that

$$Z_1 = (S_d)^{1/2} \int dR \int \prod_{(l \neq 0),\alpha} da_{l,\alpha} \, e^{-\mathcal{H}}. \tag{3.5}$$

The $l = 1$ mode, which is d-fold degenerate, may be replaced by an integral over all possible coordinates x_i $(i = 1, \ldots, d)$ of the centre of the droplet. To see this, suppose we have a specific droplet configuration $f(\eta)$. Under a rigid translation by vector x, $f(\eta) \to f_x(\eta)$; the infinitesimal form of the transformation is given in (2.8). The quantity $\int f_x(\eta)\eta_i \, d\Omega$, where η_i are the components of the unit radial vector, represents the position of the 'centre of mass' of the surface of the translated droplet. Consider the following identity for a given field configuration f:

$$\int d^d x \, \left| \det\left(\frac{\partial}{\partial x_k} \int f_x(\eta)\eta_i \, d\Omega\right)\right| \delta^d\left(\int f_x(\eta)\eta_i \, d\Omega\right) = 1. \tag{3.6}$$

The δ-function gives a contribution in $\int d^d x$ only when the centre of the translated

droplet is at the origin; the Jacobian factor from the δ function is to be evaluated at this value of x. We now substitute this identity into the integrand in (3.5) and exchange the orders of integration over x and $f(\eta)$. In the integral over f, we make the change of variable $f(\eta) \rightarrow f_{-x}(\eta)$. The Hamiltonian $\mathcal{H}(f)$ (cf §2.3) and the measure (cf appendix 1) are invariant under this transformation. The remaining factors from (3.6) simplify since $f_x(\eta) \rightarrow f_{x-x}(\eta) = f(\eta)$ and we obtain

$$Z_1 = \int d^d x \, (S_d)^{1/2} \int dR \int \prod_{l(\neq 0),\alpha} da_{l,\alpha} \, \delta^d \left(\int f(\eta) \eta_i \, d\Omega \right)$$

$$\times \left| \det \left(\frac{\partial}{\partial x_k} \int f_x(\eta) \eta_i \, d\Omega \right) (x = 0) \right| e^{-\mathcal{H}}. \tag{3.7}$$

The change of order of integration means that the δ function now constrains the function $f(\eta)$: it sets the $l = 1$ component to zero. A normalised basis for the $l = 1$ spherical harmonics is $(d/S_d)^{1/2} \eta_i$ $(i = 1, \ldots, d)$. Hence

$$\delta^d \left(\int f(\eta) \eta_i \, d\Omega \right) = (d/S_d)^{d/2} \delta^d (a_{1,\alpha}) \tag{3.8}$$

We must also evaluate the final Jacobian factor in (3.7). Since the derivative of $f_x(\eta)$ is to be evaluated at $x = 0$, we need only the infinitesimal form (2.8) of the translated $f(\eta)$; the required factor then follows as

$$\left| \det \left(\int [\eta_i \eta_k + (R + f)^{-1} \eta_i \eta_j L_{kj} f] \, d\Omega \right) \right|$$

$$= \left(\frac{S_d}{d} \right)^d \left| \det \left(\delta_{ik} + \frac{d}{S_d} \int (R + f)^{-1} \eta_i \eta_j (L_{kj} f) \, d\Omega \right) \right|. \tag{3.9}$$

Substituting (3.8) and (3.9) into (3.7) yields

$$Z_1 = (S_d)^{(d+1)/2} \, d^{-d/2} \int d^d x \int dR \int \prod_{l,\alpha}' da_{l,\alpha} \, e^{-\mathcal{H}}$$

$$\times \left| \det \left(\delta_{ik} + \frac{d}{S_d} \int (R + f)^{-1} \eta_i \eta_j (L_{kj} f) \, d\Omega \right) \right|. \tag{3.10}$$

This expression is the starting point for a standard perturbation treatment. The prime on the measure means that the $l = 0$ and $l = 1$ components are excluded from f. The integral $\int d^d x$ is just the volume V available to the droplet; it has replaced the integral over the $l = 1$ components of f.

3.2. Perturbative calculation

We now consider the perturbative evaluation of the integral over the modes with $l \geq 2$. In the expansion (3.1) for $\mathcal{H}(f)$, the first term is given in (3.2), the second term (linear in f) is zero because the $l = 0$ mode is now excluded from f, and the third term gives a Gaussian integral. Higher-order terms in f give contributions of $O(T_0)$ to Z_1 and are neglected in the one-loop calculation which we give here. The Jacobian

factor must also be expanded:

$$\left| \det\left(\delta_{ik} + \frac{d}{S_d}\int (R+f)^{-1}\eta_i\eta_j(L_{kj}f)\,d\Omega\right)\right|$$

$$= \exp\left[\text{Tr}\ln\left(\delta_{ik} + \frac{d}{S_d}\int (R+f)^{-1}\eta_i\eta_j(L_{kj}f)\,d\Omega\right)\right]$$

$$= \exp\left(\frac{d}{S_d}\int (R+f)^{-1}\eta_i\eta_j(L_{ij}f)\,d\Omega + O(f^2)\right)$$

$$= \exp[O(f^2)] \tag{3.11}$$

since $L_{ij} = -L_{ji}$. This $O(f^2)$ term differs from the quadratic term in $\mathcal{H}(f)$ because the latter carries a factor $1/T_0$. If we evaluate the contribution to Z_1 of (3.11) in perturbation theory using the free propagator from $\mathcal{H}(f)$, then we see that (3.11) is unity to within correction terms $O(T_0)$; the Jacobian factor is required only at two-loop and higher order.

Hence, in perturbation theory, expression (3.10) becomes

$$Z_1 = (S_d)^{(d+1)/2}d^{-d/2}\int d^dx\int dR\,\exp(-S_d R^\varepsilon/T_0)$$

$$\times \exp\left[-\tfrac{1}{2}\sum_{l=2}^{\infty}\nu_l(d)\ln\left(\frac{l(l+\varepsilon-1)+\varepsilon(\varepsilon-1)}{T_0 R^{2-\varepsilon}2\pi}\right)\right][1+O(T_0)]. \tag{3.12}$$

The contribution of the Gaussian integral is given by the sum over $l \geq 2$. The degeneracy factor $\nu_l(d)$ is given in (2.9) and we have made the substitution (3.3) in the Gaussian integral.

The remainder of this section is concerned with the evaluation of the sum in (3.12). The first question is the convergence of the sum. From (2.9) the asymptotic form of $\nu_l(d)$ for large l is

$$\nu_l(d) \sim (2/\Gamma(d-1))l^{\varepsilon-1}.$$

Hence we see that the sum in (3.12) diverges for any $\varepsilon > 0$, i.e. for all dimensions $d > 1$. This is a direct consequence of the continuum limit implicit in the form of the Hamiltonian (2.11). To make the sum meaningful we must regularise the theory in some way. The most physically acceptable way of doing so is to restore the cut-off effects afforded by the underlying lattice. This procedure is hard to implement in this case: it cannot be done simply by imposing a cut-off on the sum over l as this would break the symmetry under rigid translations. One possible alternative is to work within the droplet framework which emerges from the Landau–Ginzburg model with *finite* well depth, reflected in a finite mass parameter μ. The droplet interface is then smeared over distances $O(\mu^{-1})$, producing an effectively translation-invariant cut-off on l of $O(\mu R)$ where R is the droplet scale size. We do not pursue this approach here but note that the regularising effect of μ is evident in the calculations reported in appendix 1.

We shall define the sum in (3.12) by dimensional regularisation, which means that we must envisage continuing the sum from *negative* values of ε, for which it is convergent. Dimensional regularisation has been successfully applied to a number of similar problems involving spherical harmonic functions in d dimensions (McKane

and Wallace 1978, Drummond and Shore 1979, McKane 1979). Here we shall see that the sum in (3.12) has a Laurent expansion with a simple pole at $\varepsilon = 0$. The perturbative corrections omitted from (3.12) turn out to be at most $O(\varepsilon)$ (after renormalisation: cf § 3.3) which is, therefore, the order to which we now evaluate the sum.

We begin by noting from (2.9) that

$$\sum_2^\infty \nu_l(d) = \lim_{L\to\infty} \sum_{l=2}^L \left(\frac{\Gamma(l+d)}{\Gamma(d)\Gamma(l+1)} - \frac{\Gamma(l+d-2)}{\Gamma(d)\Gamma(l-1)} \right)$$

$$= \lim_{L\to\infty} \left(-1-d + \frac{\Gamma(L+d)}{\Gamma(d)\Gamma(L+1)} + \frac{\Gamma(L+d-1)}{\Gamma(d)\Gamma(L)} \right)$$

$$= -1-d, \tag{3.13}$$

the sum converging for $d < 1$. Hence one sum in the Jacobian factor is given by

$$\exp\left(\tfrac{1}{2} \sum_{l=2}^\infty \nu_l(d) \ln(T_0 R^{2-\varepsilon} 2\pi) \right) = (2\pi T_0 R^{2-\varepsilon})^{-(2+\varepsilon)/2}. \tag{3.14}$$

The remaining sum is

$$\Sigma_2 = \sum_2^\infty \nu_l(d) \ln[l(l+\varepsilon-1) + \varepsilon(\varepsilon-1)]$$

$$= \frac{2}{\Gamma(d-1)} \sum_2^\infty \frac{\Gamma(l+\varepsilon-1)}{\Gamma(l)} \ln[l(l-1)] + O(\varepsilon) \tag{3.15}$$

using (2.9) for $\nu_l(d)$ and simplifying for ε small. In order to evaluate this sum we need the expansion

$$\Gamma(l+\varepsilon)/\Gamma(l) = l^\varepsilon[1 - \varepsilon(\tfrac{1}{2}l^{-1} + O(l^{-2})) + O(\varepsilon^2/l)].$$

Then

$$\sum_2^\infty \frac{\Gamma(l+\varepsilon-1)}{\Gamma(l)} \ln l = \sum_1^\infty l^{\varepsilon-1} \ln l + O(1). \tag{3.16}$$

Similarly

$$\sum_2^\infty \frac{\Gamma(l+\varepsilon-1)}{\Gamma(l)} \ln(l-1) = \sum_1^\infty l^{\varepsilon-1} \ln l + O(\varepsilon). \tag{3.17}$$

Now the Riemann zeta-function is defined by

$$\zeta(z) = \sum_{n=1}^\infty n^{-z}$$

and hence

$$\sum_1^\infty l^{\varepsilon-1} \ln l = -\zeta'(1-\varepsilon).$$

Further, $\zeta(z)$ has a simple pole at $z = 1$ (Gradshteyn and Ryzhik 1965, § 9.536):

$$\zeta(z) = (z-1)^{-1} + O(1)$$

and hence

$$\sum_1^\infty l^{\varepsilon-1} \ln l = \varepsilon^{-2} + O(1). \tag{3.18}$$

Collecting together the results (3.15)–(3.18) gives the remaining sum to the required accuracy. The corresponding factor in Z_1 is given by

$$\exp(-\tfrac{1}{2}\textstyle\sum_2) = \exp(-2/\Gamma(d)\varepsilon)[1 + O(\varepsilon)]. \tag{3.19}$$

Substituting expressions (3.14) and (3.19) into (3.12) gives

$$Z_1 = (S_d/2\pi)^{(d+1)/2} d^{-d/2} \int d^d x \int dR\,(T_0 R^{2-\varepsilon})^{-(2+\varepsilon)/2}$$

$$\times \exp[-(S_d R^\varepsilon T_0^{-1} + 2/\varepsilon\,\Gamma(d))][1 + O(T_0, \varepsilon)] \tag{3.20}$$

for the single-droplet partition function. By evaluating the Gaussian integral only up to terms of order 1 we neglect corrections of order ε; corrections of order T_0 arise at next order in perturbation theory.

3.3. Renormalisation

The argument of the exponential in (3.20) exposes the limitations of the bare perturbation theory calculation. The classical contribution $S_d R^\varepsilon T_0^{-1}$ is corrected by a factor $[1 + 2T_0 R^{-\varepsilon}/(\varepsilon\,\Gamma(d)S_d)]$ from the Gaussian integral (the one-loop contribution). The correction factor exhibits two features which are generic to the perturbation expansion in the coupling T_0, which has dimension (length)$^\varepsilon$. First, the effective dimensionless expansion parameter is $T_0 R^{-\varepsilon}$. Second, ultraviolet divergences from large l in sums over l and α in Feynman graphs contribute up to a factor ε^{-1} for each order in perturbation theory. The result (3.20) is therefore inadequate for our purposes on two grounds; the effective expansion parameter for $\varepsilon > 0$ is unboundedly large for small droplets ($R \to 0$) and the factors ε^{-1} from each loop in perturbation theory have to be eliminated.

The situation is similar to that which arises in describing critical behaviour in $(4-\varepsilon)$ dimensions in the Landau–Ginzburg model. The dimensionless bare anharmonic coupling is $g(\xi)^{4-d}$, which becomes unboundedly large as the correlation length ξ diverges (for $d < 4$); if one relies on dimensional regularisation without an explicit cut-off, there are also factors of $(4-d)^{-1}$ for each loop integral. Renormalisation theory (Amit 1978) resolves both of these problems by introducing a dimensionless renormalised coupling which absorbs the $(4-d)^{-1}$ factors and which, as the length scale ξ diverges, increases to a *limiting* value of order $(4-d)$, the RG fixed point.

We are going to follow the same procedure to extend the validity of the perturbation expansion (3.20). We shall introduce a dimensionless renormalised temperature parameter $T(R)$ which eliminates the ε^{-1} factors and which, as the scale size R of the droplet decreases to zero, increases to a limiting value which is a RG fixed point. The difference between the procedure here and that encountered in $4-\varepsilon$ dimensions is that here we are controlling the effect of *small-scale* fluctuations ($R \to 0$) and will need an ultraviolet stable fixed point to do so; in the Landau–Ginzburg approach we are controlling *large-scale* fluctuation ($\xi \to \infty$) and need an infrared stable fixed point. The ultraviolet stable fixed point required for T emerges naturally in $1+\varepsilon$ dimensions ($\varepsilon > 0$): it turns out to be the critical temperature at which the droplet density is

sufficient to destroy the long-range order. The different stability properties for T and the Landau–Ginzburg coupling g are essential because T is a relevant variable and g is irrelevant. The calculations themselves follow the renormalisation discussed for the planar interface (Wallace and Zia 1979, Forster and Gabriunas 1981, David 1981) and are in the same spirit as the ε expansions in $2 + \varepsilon$ dimensions for Heisenberg models (see e.g. Brézin and Zinn-Justin 1976a, b, Brézin *et al* 1976a, b and references therein).

Because the integrand in (3.20) is exponentially small in T_0, care is required at several points in the renormalisation. In the context of dimensional regularisation the bare coupling has an expansion in the renormalised coupling which is usually written in the form (Amit 1978)

$$T_0 \mu^\varepsilon = T(\mu) + AT^2(\mu) + O(T^3) \tag{3.21}$$

where μ is some arbitrary momentum scale and A contains a pole term in ε^{-1} in order to remove the ε^{-1} terms which appear in perturbation theory. If A and higher-order coefficients contain *only* negative powers of ε, then we have a minimal subtraction scheme. Equation (3.21) then defines uniquely a renormalised coupling at momentum scale μ, for which standard RG equations follow. With regard to expression (3.20), it is more natural to define a renormalised coupling at length scale R. A minimal subtraction renormalisation is

$$T_0 R^{-\varepsilon} = T_{\mathrm{MS}}(R) + \varepsilon^{-1} T_{\mathrm{MS}}^2(R) + O(T^3). \tag{3.22}$$

Substituting into (3.20) we obtain

$$Z_1 = (S_d/2\pi)^{(d+1)/2} d^{-d/2} \int \mathrm{d}^d x \int \mathrm{d}R \, R^{-(2+\varepsilon)} (T_{\mathrm{MS}}(R))^{-(2+\varepsilon)/2}$$

$$\times \exp\left[-\left(\frac{S_d}{T_{\mathrm{MS}}(R)} + \frac{2}{\varepsilon \Gamma(d)} - \frac{S_d}{\varepsilon} \right) \right] [1 + O(\varepsilon, T_{\mathrm{MS}}(R))]. \tag{3.23}$$

The renormalisation has clearly eliminated the problem of the ε^{-1} factor since the argument of the exponential is finite as $\varepsilon \to 0$.

The remaining problem in (3.23) is to study the R dependence of $T_{\mathrm{MS}}(R)$ for small R. This is given by the RG equation obtained from (3.22) by differentiation holding T_0 fixed:

$$R(\partial T_{\mathrm{MS}}/\partial R) \equiv \beta(T_{\mathrm{MS}}) = -\varepsilon T_{\mathrm{MS}} + T_{\mathrm{MS}}^2 + O(T_{\mathrm{MS}}^3). \tag{3.24}$$

This RG equation has two fixed points:
  (a) $T = 0$: infrared stable, $T_{\mathrm{MS}}(R) \to 0$ as $R \to \infty$;
  (b) $T_c = \varepsilon + O(\varepsilon^2)$: ultraviolet stable, $T_{\mathrm{MS}}(R) \to T_c$ as $R \to 0$.
The existence of this second fixed point allows us to achieve our second aim; as $R \to 0$, the new expansion parameter $T_{\mathrm{MS}}(R)$ increases, but only to a finite value of $O(\varepsilon)$. This behaviour is to be contrasted with the bare expansion in $T_0 R^{-\varepsilon}$, which is useless for $R \to 0$.

The explicit dependence of the expansion parameter T_{MS} upon the scale size R is obtained by integrating (3.24). If we neglect $O(T^3)$ from two-loop diagrams, the result may be written in the form

$$T_{\mathrm{MS}}(R)[1 - T_{\mathrm{MS}}(R)/\varepsilon]^{-1} R^\varepsilon = \text{constant} \equiv c_0 \xi^\varepsilon \tag{3.25a}$$

where c_0 is a dimensionless constant and ξ an R-independent constant with the

dimensions of length. This length scale ξ plays in every way the role of the correlation length.

In the first instance we see that for scale sizes R small compared with ξ, $T_{MS}(R)$ approaches the ultraviolet stable fixed point value $T_c = \varepsilon + O(\varepsilon^2)$: this result expresses the fact that criticality is evident on any length scale small compared with the correlation length.

Secondly we observe that, for $R \gg \xi$, equation (3.25a) implies that $T_{MS}(R) \approx c_0 (R/\xi)^{-\varepsilon}$; accordingly the surface energy for droplets of such scale sizes is $c_0^{-1} S_d (R/\xi)^\varepsilon$, the classical result for a droplet of radius R and surface tension $\sigma = T_0^{-1} \approx c_0^{-1} \xi^{-(d-1)}$. The latter observation bears out the expectations of scaling theory (Fisk and Widom 1969).

Thirdly, noting that ξ is (by construction) an R-independent length, we see that we may replace R in (3.25) by some fixed length scale L_0 (to be thought of as a minimum droplet size, or 'effective' lattice spacing). Identifying $T_{MS}(L_0)$ as the true system temperature and the regime $T_{MS}(L_0) \approx T_c$ as the critical region, we then find for ξ the expected power law behaviour

$$\xi \approx L_0 (T_c/c_0)^{1/\varepsilon} [1 - T_{MS}(L_0)/T_c]^{-\nu} \tag{3.25b}$$

where

$$\nu = 1/\varepsilon. \tag{3.25c}$$

Finally we remark that our identification of ξ as a measure of the bulk correlation length is also borne out by a study of the $d = 1$ limit of our theory (appendix 2) which yields a correlation length with the same temperature dependence as that prescribed by kink-based theories of one-dimensional Landau–Ginzburg models.

Implicit in the foregoing observations are three potentially disconcerting discoveries which merit further comment. The parameter T_0 introduced in the Hamiltonian (2.11) is *not* a measure of the system temperature; an effective lattice spacing, L_0, must be introduced by fiat; it is $T_{MS}(L_0)$ which plays the role of the system temperature. These peculiarities are interrelated corollaries of the fact that, within our chosen framework of dimensional regularisation, the Hamiltonian (2.11) has no explicit short-distance cut-off. Analogous circumstances are encountered in the dimensionally regularised treatment of the Landau–Ginzburg model near dimension $d = 4$ (see e.g. Brézin *et al* 1973).

Now it would appear that, by solving (3.25a) for $T_{MS}(R)$ and substituting into (3.23), we obtain the required expression for Z_1. However, this procedure is incorrect: the overall amplitude is obtained correctly only if the R dependence of $T_{MS}(R)$ is given to two loops, and not just one as in (3.25a). In particular, as $R \to 0$

$$\exp[-S_d/T_{MS}(R)] \approx \exp(-S_d/T_c).$$

Since $T_c = \varepsilon + O(\varepsilon^2)$, we need the $O(\varepsilon^2)$ term in T_c to give the amplitude correct to $O(1)$. This remark is a potential source of concern because although T_c is universal to order ε, it is not a universal quantity to $O(\varepsilon^2)$, since it depends on the precise definition of the renormalised T. We complete this section by giving the systematic two-loop renormalisation in such a way as to make explicit the universal character of Z_1. In so doing we shall invoke the known two-loop form of the β function for the almost *planar* interface (Wallace and Zia 1979)

$$\beta(T) = -\varepsilon T + AT^2 + \tfrac{1}{2}BT^3 + O(T^4) \tag{3.26a}$$

with

$$A = 1 + O(\varepsilon), \qquad B = 1 + O(\varepsilon). \tag{3.26b}$$

This form clearly coincides with that prescribed in (3.24), to the *one*-loop order of approximation inherent in the latter. In fact, if the nearly-spherical-interface theory is to capture correctly the (presumably universal) two-loop corrections to the exponent ν (cf (3.28c) below), the corresponding β function for the droplet problem must coincide with equations (3.26) at *two* loops also.

Proceeding on this basis, we observe first that the correlation length may be introduced systematically as the R-independent length scale. Thus, writing $\xi = Rg(T(R))$ and requiring that $d\xi/dR = 0$, one finds that the correlation length is prescribed up to an overall amplitude:

$$\xi = R \exp\left(-\int^{T(R)} dT'/\beta(T')\right). \tag{3.27}$$

Substituting (3.26) into (3.27) and integrating, we obtain the two-loop generalisations of (3.25a, b, c):

$$T(R)[1 - T(R)/T_c]^{-\varepsilon\nu}[1 + O(\varepsilon T)] = c_0(\xi/R)^\varepsilon, \tag{3.28a}$$

$$\xi \approx L_0(T_c/c_0)^{1/\varepsilon}[1 - T(L_0)/T_c]^{-\nu}, \tag{3.28b}$$

$$\nu^{-1} = \beta'(T_c) = \varepsilon + \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3). \tag{3.28c}$$

We note that the non-universal $O(\varepsilon)$ correction to A cancels to give a universal form for ν.

We are now in a position to eliminate T_0 in favour of a renormalised T in a manifestly universal way at this order in perturbation theory. The form of the β function (3.26a) prescribes, for T_0, an expansion in powers of $T(R)$ of the form

$$T_0 R^{-\varepsilon} = T(R)[1 + (\varepsilon\nu/T_c)T(R)] + O(T^3). \tag{3.29a}$$

Hence

$$R^\varepsilon/T_0 = 1/T(R) - \varepsilon\nu/T_c + O(T(R)). \tag{3.29b}$$

Substituting this expression into (3.20), we obtain

$$Z_1 = (S_d/2\pi)^{(d+1)/2} d^{-d/2} \int d^d x \int dR\, R^{-(2+\varepsilon)}(T(R))^{-(2+\varepsilon)/2}$$

$$\times \exp\left[-S_d\left(\frac{1}{T(R)} - \frac{\varepsilon\nu}{T_c}\right) - \frac{2}{\varepsilon\Gamma(d)}\right][1 + O(\varepsilon, T(R))]. \tag{3.30}$$

In conjunction with expressions (3.28a) for $T(R)$ and (3.28c) for ν this is the key result of this section. Since $T_c = \varepsilon + O(\varepsilon^2)$, the $1/\varepsilon$ pole in the exponential is of course cancelled by the renormalisation as before. Although T_c appears in (3.30) and in (3.28a), its numerical value is required only at order ε. For example, as $R \to 0$, the term in the exponential in (3.30) becomes

$$\frac{1}{T(R)} - \frac{\varepsilon\nu}{T_c} \approx \frac{1}{T_c}(1 - \varepsilon\nu) = \tfrac{1}{2}\frac{\varepsilon}{T_c} + O(\varepsilon) = \tfrac{1}{2} + O(\varepsilon) \tag{3.31}$$

which is manifestly universal.

4. Single-droplet properties

In this section we establish a number of properties of isolated droplets which follow from the key results of § 3, are of interest in their own right, and will prove useful in the development of the theory of many-droplet configurations initiated in § 5. We shall begin by showing that the single-droplet partition function (3.30) may be expressed in a universal scaling form which, we shall ultimately see, is instrumental in controlling the mean population of droplets of a given scale size. We shall then examine the manner in which surface fluctuations modify the volume and surface area of a droplet. Finally we shall establish the behaviour of the radial variance of droplets of a given scale size, giving an explicit statistical measure of the extent to which the droplet shapes deviate from spherical.

4.1. Scaling form of the single-droplet partition function

The single-droplet partition function (3.30) may be cast in a particularly useful form:

$$Z_1 = V \int dR \, v_0^{-1}(R) \psi(R, \xi). \tag{4.1}$$

We have used the fact that, in (3.30), the integral with respect to x extends over all possible positions of the centre of the reference droplet; the integrand is independent of x and so the integral gives the volume V of the available embedding space. We have also introduced the volume $v_0(R)$ of a hypersphere of radius R,

$$v_0(R) = d^{-1} S_d R^d \tag{4.2}$$

which, we shall see below (cf (4.7)), is effectively the mean volume of droplets of scale size R. The function $\psi(R, \xi)$ is defined by

$$\psi(R, \xi) = 2\pi^{-1} R^{-1} (T(R))^{-(2+\varepsilon)/2}$$

$$\times \exp\left[-S_d \left(\frac{1}{T(R)} - \frac{\varepsilon \nu}{T_c} \right) - \frac{2}{\varepsilon \Gamma(d)} \right] [1 + O(\varepsilon, T(R))] \tag{4.3}$$

where the prefactor has been simplified by the neglect of corrections of order ε. Since $T(R)$ is a function of R/ξ only (cf (3.28a)), equation (4.3) has the scaling form

$$\psi(R, \xi) = R^{-1} \tilde{\psi}(R/\xi). \tag{4.4a}$$

The limiting behaviour of the function $\tilde{\psi}$ will prove to be important. For large values of its argument z the function $\tilde{\psi}(z)$ approaches zero exponentially fast:

$$\tilde{\psi}(z) \approx \psi^{(\infty)} z^{\varepsilon(2+\varepsilon)/2} \exp(-S_d z^\varepsilon / c_0). \tag{4.4b}$$

For small values of z the function has a power series expansion of the form

$$\tilde{\psi}(z) = \sum_n \psi_n z^{n/\nu} \tag{4.4c}$$

where, in particular,

$$\psi_0 = \lim_{z \to 0} \tilde{\psi}(z) = 2\pi^{-1} \varepsilon^{-(2+\varepsilon)/2} \exp(-1 - 2C - 2/\varepsilon) \tag{4.4d}$$

with $C = 0.577 \ldots$ the Euler constant.

The physical significance of the function ψ, and the parameter ψ_0 in particular, will become clear in the course of § 5.

4.2. Mean volume of a single droplet

The mean volume of a droplet of scale size R may be determined by evaluating the average of the volume operator (2.3) in the ensemble prescribed by the single-droplet partition function (2.12). Expanding (2.3) in powers of the field f we find

$$v(R) = \langle \mathcal{V} \rangle_R = d^{-1} \int d\Omega \, R^d \langle 1 + dR^{-1}f(\eta) + \tfrac{1}{2}\varepsilon dR^{-2}f^2(\eta) + O(f^3) \rangle_R. \tag{4.5}$$

We now make the decomposition into spherical harmonics prescribed by (2.2). The Jacobian factor associated with the transformation again does not enter at the order of calculation considered here. The contribution of the $l = 1$ modes may be dropped since the volume is invariant under the rigid translations (2.8). The contribution of the $l = 0$ mode is also suppressed by the constraint that the average in (4.5) be taken for a prescribed R. Noting that $\int d\Omega \, f(\eta)$ then vanishes identically, and utilising the free propagator of the expansion (3.1), we then find from (4.5)

$$v(R) = d^{-1}S_d R^d \left(1 + \frac{\varepsilon d}{2S_d} T_0 R^{-\varepsilon} \sum_{2}^{\infty} \nu_l(d)[l(l + \varepsilon - 1) + \varepsilon(\varepsilon - 1)]^{-1} + O(T_0^2) \right). \tag{4.6}$$

We note that the sum in (4.6) converges: by dimensional analysis, the volume operator $\mathcal{V}$ requires no renormalisation beyond the replacement of $T_0 R^{-\varepsilon}$ by $T(R)$ (given in (3.28)) to eliminate subdivergences. The sum can be performed exactly in the limit $\varepsilon \to 0$: using expression (2.9) for $\nu_l(d)$ one obtains

$$\sum_{2}^{\infty} \nu_l(d)[l(l + \varepsilon - 1) + \varepsilon(\varepsilon - 1)]^{-1} = (\varepsilon/\Gamma(d))(1 + O(\varepsilon)).$$

Substituting into (4.6), replacing $T_0 R^{-\varepsilon}$ by $T(R)$ at this order in perturbation theory, and using $S_d = 2 + O(\varepsilon)$, one obtains

$$v(R) = d^{-1}S_d R^d [1 + \tfrac{1}{4}\varepsilon^2 T(R)(1 + O(\varepsilon)) + O(T^2(R))] = v_0(R)[1 + O(T^2(R))]. \tag{4.7}$$

Since, below the critical temperature, $T(R)$ is bounded above by $T_c = O(\varepsilon)$, equation (4.7) shows that the corrections to the hyperspherical volume, (4.2), due to droplet shape fluctuations are at least of order ε^2. (In fact, we expect that the $O(T^2(R))$ terms also contain a multiplicative factor $O(\varepsilon)$.) Within the context of the present work the difference between $v(R)$ and $v_0(R)$ can thus be safely neglected.

4.3. Mean surface area of a single droplet

The behaviour of the surface area has been considered in the context of the almost planar interface by David (1981), who shows that the necessary renormalisation of the surface operator implies that a surface of scale size $L \gg \xi$ indeed scales as L^{d-1}, whereas a surface of scale size $L \ll \xi$ scales as $L^{1/\nu}$ and thus has a *fractal* character (a point to which we shall return in § 8). The same surface properties hold for a droplet of scale size R. The perturbative calculation of the mean of the surface area operator $\mathcal{A}$ in (2.7) can be done directly, as we did in § 4.2 for the volume operator

$\mathcal{V}$, or from the one-droplet partition function Z_1, noting that

$$\langle \mathcal{A} \rangle = T_0^2 \, \partial \ln Z_1 / \partial T_0. \tag{4.8}$$

The expression (3.20) for Z_1 gives

$$\langle \mathcal{A} \rangle_R = S_d R^\varepsilon \left(1 - \frac{2+\varepsilon}{2S_d} T_0 R^{-\varepsilon} + \mathrm{O}(T_0^2 R^{-2\varepsilon}) \right), \tag{4.9}$$

where again we consider the scale size R of the droplet as fixed.

The result (4.9) in bare perturbation theory contains an apparent paradox; the correction term $\mathrm{O}(T_0 R^{-\varepsilon})$ has a negative coefficient suggesting the nonsensical result that fluctuations *decrease* the surface area. In order to make sense of (4.9) one must look carefully at the renormalisation. In contrast to the volume operator $\mathcal{V}$, dimensional analysis implies that the surface area operator $\mathcal{A}$ *does* require a multiplicative renormalisation. Specifically, since T_0 has a singular expansion in $T(R)$ and $\ln Z_1$ has a finite expansion (no ε^{-1}) in $T(R)$, then $\langle \mathcal{A} \rangle_R$ in (4.8) necessarily has an expansion involving ε^{-1} poles which require renormalisation. We must thus find a dimensionless renormalisation factor for the bare area $\langle \mathcal{A} \rangle_R$ which removes these poles in ε, leaving a finite result. Introducing again the cut-off length scale L_0 at which the renormalised coupling has the value $T \equiv T(L_0)$ and noting that

$$L_0^\varepsilon T^2 (\partial \ln Z_1 / \partial T) \equiv L_0^\varepsilon T^2 (\partial T_0 / \partial T) \partial \ln Z_1 / \partial T_0$$

is an area which is automatically finite, we can *define* the renormalised mean area $a(R, L_0, T)$ by

$$a(R, L_0, T) = (L_0^\varepsilon T^2 T_0^{-2} \, \partial T_0 / \partial T) \langle \mathcal{A} \rangle_R. \tag{4.10}$$

The bare area $\langle \mathcal{A} \rangle_R$ depends only on T_0 and R. Differentiating with respect to L_0 at fixed T_0 and R, and using the chain rule as usual, we obtain the RG equation for the renormalised surface area:

$$[L_0(\partial / \partial L_0) + \beta(T)\partial / \partial T + \gamma_a(T)] a(R, L_0, T) = 0 \tag{4.11}$$

where

$$\gamma_a(T) = -\varepsilon - 2\beta(T)/T + \mathrm{d}\beta(T)/\mathrm{d}T. \tag{4.12}$$

As usual, $\gamma_a(T)$ comes from the renormalising scale factor in (4.10). This equation for a droplet surface of scale size R is equivalent to the equation for the planar interface derived by David (1981) (with changes in sign conventions in $\gamma(T)$ and $\beta(T)$).

The analysis of (4.11) follows standard RG methods. The values of (4.12) at the fixed points play an important role in determining the scaling behaviour with R: using the expressions in (3.26) and (3.28) we find

$$\gamma_a(0) = 0, \qquad \gamma_a(T_c) = -\varepsilon + 1/\nu. \tag{4.13}$$

Equation (4.11) is integrated in the usual way:

$$a(R, L_0, T) = \exp \left(\int_{L_0}^L \gamma_a(T(L')) \frac{\mathrm{d}L'}{L'} \right) a(R, L, T(L)). \tag{4.14}$$

In order to evaluate $a(R, L_0, T)$ in perturbation theory, the appearance of $T_0 R^{-\varepsilon}$ in (4.9) makes it clear that we should choose the free length scale L equal to R, so that

the RHS of (4.14) can be evaluated perturbatively. Using the expansion

$$T_0 L_0^{-\varepsilon} = T[1 + (\varepsilon \nu / T_c)T + O(T^2)]$$

from (3.29), one finds

$$L_0^\varepsilon T^2 T_0^{-2} \partial T_0 / \partial T = (1 + \varepsilon \nu T / T_c)^{-2} (1 + 2\varepsilon \nu T / T_c) + O(T^2)$$

$$= 1 + O(T^2). \tag{4.15}$$

Combining (4.9), (4.10) and (4.15), we obtain from (4.14)

$$a(R, L_0, T) = \exp \left(\int_{L_0}^{R} \gamma_a(T(L')) \frac{1}{L'} \, dL' \right) S_d R^\varepsilon \left(1 - \frac{2 + \varepsilon}{2 S_d} T(R) + O(T^2(R)) \right). \tag{4.16}$$

This is the key result for the surface area. Some remarks are in order. The renormalising factor in (4.10) in fact is required perturbatively only at $O(T^2)$ according to (4.15). Correspondingly the bare perturbative calculation (4.9) does not *appear* to acquire renormalisation at order T; it certainly *is* required at $O(T^2)$ to create the anomalous dimension $\gamma_a(T_c) = O(\varepsilon^2)$. If $R \gg \xi$, there is asymptotically no R dependence in the exponential prefactor in (4.16) since $\gamma_a(T(L')) \sim L'^{-\varepsilon}$ for $L' \gg \xi$. Thus the surface area of droplets of scale size $R \gg \xi$ scales as the conventional area R^ε. For $L' \ll \xi$, however, $\gamma_a(T(L')) \simeq \gamma_a(T_c) = -\varepsilon + 1/\nu$, implying, for $R \ll \xi$, the scaling behaviour

$$a(R, L_0, T) = S_d R^\varepsilon (R/L_0)^{1/\nu - \varepsilon} [1 + O(T(R))]. \tag{4.17}$$

Thus droplets of scale sizes less than the correlation length have a surface area with an anomalous dimension. Clearly we cannot extrapolate L_0 to zero, since the surface area of any finite droplet would then be infinite; this just reflects the fact that the surface operator requires renormalisation.

Finally, we note that the negative coefficient of the $O(T(R))$ correction in (4.9) can now be seen to be entirely meaningful. As R decreases $T(R)$ increases and the negative coefficient implies that the surface area decreases more rapidly than just by the scale factor R^ε. However, in decreasing R to $R - \Delta R$, say, we are *losing* surface fluctuations of wavelength between $R - \Delta R$ and R and hence must expect the surface area to decrease by more than $(R - \Delta R)^\varepsilon / R^\varepsilon$.

4.4. Radial variance for a single droplet

The extent to which the typical droplet of scale size R deviates from spherical is characterised by the radial variance $\langle f^2 \rangle_R$, which may be regarded as a measure of the droplet ramification in the terminology of Domb (1976). This quantity has already been calculated in the course of § 4.2. The calculation to obtain (4.7) from (4.6) yields

$$\langle f^2 \rangle_R = S_d^{-1} T_0 R^{2-\varepsilon} \sum_2^\infty \nu_l(d) [l(l + \varepsilon - 1) + \varepsilon(\varepsilon - 1)]^{-1} + O(T_0^2)$$

$$= \tfrac{1}{2} R^2 [T(R)\varepsilon (1 + O(\varepsilon)) + O(T^2(R))]. \tag{4.18}$$

As in all previous calculations, the crossover of $T(R)$ occurring between the ranges $R \gg \xi$ and $R \ll \xi$ controls the behaviour of $\langle f^2 \rangle_R$. For $R \gg \xi$, with $T(R) \simeq c_0 (\xi/R)^\varepsilon$, we see

$$\langle f^2 \rangle_R \simeq \tfrac{1}{2} c_0 \varepsilon R^{2-\varepsilon} \xi^\varepsilon. \tag{4.19a}$$

This is the standard dependence due to the classical wandering of the interface. For $R \ll \xi$, $T(R) \simeq T_c$ and

$$\langle f^2 \rangle_R \simeq \tfrac{1}{2}\varepsilon^2 R^2. \tag{4.19b}$$

Hence we see again the scale invariance of droplets small compared with the correlation length; the fractional variance $\langle f^2 \rangle_R / R^2$ is independent of R.

5. Multi-droplet configurations

Thus far we have focused on the characteristics of isolated droplets. We have seen that certain critical-point properties of the bulk Ising system (specifically, the existence of a critically diverging correlation length, and the associated exponent ν) are reflected in the form of the spectrum of single-droplet configurations. In this section we initiate the second part of our programme which is concerned with multi-droplet configurations. We shall develop a theory of the partition function of a many-droplet assembly. In the process we shall identify a procedure by which one can generate a multi-droplet representation of the canonical ensemble of configurations in an Ising system. We shall then utilise this ensemble to determine further critical observables (in particular, the order parameter and its exponent β) whose behaviour we shall again trace to specific features of the underlying configurations. In the course of our arguments we shall provide the justification for the assumption implicit in our earlier analysis (cf § 2.1), that one can meaningfully speak of the properties of effectively *isolated* droplets even in the critical region, in low enough dimensions.

As we indicated in § 1, the character of the arguments we shall deploy in carrying out this programme will contrast sharply with those utilised in the preceding sections: they will not require the mathematical sophistication of the latter (the key results we shall need are already available in (4.1)–(4.4)) but will offer, and occasionally demand, rather more physical insight.

5.1. Preliminaries

The simple droplet picture of Ising configurations advanced in § 2 envisages a gas of a few effectively isolated 'islands' of one phase (black, say) embedded in a 'sea' of the opposite (white) phase. This picture is certainly appropriate at low temperatures. In the vicinity of the critical point, however, it is clearly wrong in two essential respects.

First, as criticality is approached, the net volume of space occupied by black islands cannot remain small on the scale of the system volume: precisely at criticality black space and white space must have *equal* volumes.

Second, and more subtly, one must recognise, following Kadanoff (1976a; see also Stauffer *et al* 1971) that within each black island one will typically find a number of white 'lakes' which, in turn, may accommodate still smaller black islands as suggested schematically in figure 2.

It is clear *a priori* (and will become clearer in the course of our explicit analysis) that the second effect is an essential ingredient of a properly scaling (and ultimately scale-invariant) theory of the critical region. Nevertheless it is the first effect which presents the basic problems: the second 'effect' (merely!) ensures that the problems encountered in characterising the configurations of the system as a whole are re-encountered in characterising the substructure of every island (i.e. at every length
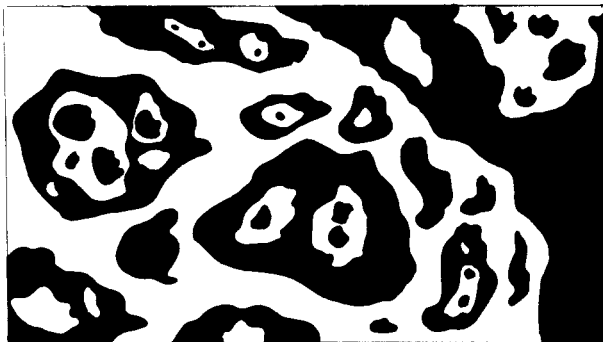
Figure 2. Nested droplets (after Kadanoff (1976a)).

scale!). For the moment, then, let us forget the (ultimately essential) fact that the black islands have such substructure and concentrate on the problems posed by the growth in their population with the approach to criticality, and the circumstances in which one might hope to find that these problems are tractable.

It is apparent that, near criticality, the black islands embedded in the white background do not realise, in any obvious way, a dilute gas of droplets. This breakdown of the dilute droplet gas approximation is, in a sense, endemic to the critical region and would appear to have inevitable and fatal consequences for any droplet-based theory of the critical point. Indeed the prognosis for the viability of a droplet theory in dimension $d = 3$ is not good. Monte Carlo studies of three-dimensional Ising systems (Müller–Krumbhaar and Stoll 1976) have long since established that the said black islands grow and merge with one another, as the temperature is raised, in such a way as to form an infinite (i.e. sample sized) connected black region, percolating through the background white region, at a temperature *below* that marking the true critical point. In such circumstances the spectrum of black islands cannot be identified with an assembly of droplets, whose (typical) largest member has a scale size (of the order of the correlation length) which diverges only *at* the critical point. If one insists on a droplet representation, the connected black regions must instead be viewed as being built from (possibly many) *overlapping* black droplets. While such a representation is always possible in principle, it is not obviously useful in practice, since it is not at all clear how to handle the statistics of overlapping (and thus, in effect, interacting) droplets. Indeed, it is in fact the 'no-overlap' approximation that forms the essential simplifying ingredient of the dilute gas picture. Recognising this, one may naturally inquire whether the (inevitable) failure of the dilute gas picture itself necessarily implies the failure of the no-overlap approximation. In fact, it does not. It is possible to envisage circumstances in which, even as the black droplet population grows critically to fill a volume comparable to the volume of the embedding space, the extent of the black droplet overlap remains small: in effect, droplet boundaries may remain dilute even when the droplets themselves do not. It turns out that this is the situation which holds in space dimensions sufficiently low that the parameter ψ_0, defined in (4.4d), may be regarded as 'small'. This assertion can be viewed as a plausible extension of the situation known to hold in one dimension where the universal configurational physics is captured by a domain wall (kink) gas model (Krumhansl and Schrieffer 1975, Zinn-Justin 1981, Bruce 1981b) which is manifestly dilute in the low-temperature (critical) regime. (Indeed, the $d \rightarrow 1$ limit of our theory can be shown to

yield the dilute kink gas results: cf appendix 2.) The assertion will be both refined and substantiated in due course. For the time being we shall presuppose its validity and proceed to show that there then emerges an explicit and analytically tractable droplet theory, whose fundamental characteristics are already implicit in the results of the preceding sections.

5.2. The multi-droplet partition function and configuration-generating procedure

In this section we shall show how one may calculate the full partition function for an Ising model given the assumption (still to be justified) that the coarse-grained configurations are built from nested black and white droplets whose boundaries are (in a sense still to be refined) dilute.

Consider, then an Ising system of volume V which, without loss of generality, we shall take to be hyperspherical with radius L. We suppose that the possible configurations of this system are defined by the set of all distinct black and white patterns formed by superimposing black and white droplets of all possible shapes and scale sizes larger than some arbitrary minimum L_0. The energy (and thence the statistical weight) of any configuration is specified by the associated total interfacial area. The partition function for the Ising system may then be written in the form

$$Z \equiv Z(L, L_0, \xi) = \sum_{\{C(L, L_0)\}} \exp\{-\mathcal{H}[C(L, L_0)]\} \tag{5.1}$$

where $\{C(L, L_0)\}$ denotes the set of configurations formed by droplets of scale size intermediate between L_0 and L, while $\mathcal{H}$ is the associated dimensionless configurational energy. Adopting a strategy familiar from RG arguments (of which the following is a variant), let us decompose the configurational sum in (5.1), by separating out the contributions made by the smallest droplets—those with scale size in the range $L_0 \rightarrow L_0 + dL_0$. Explicitly we write

$$Z(L, L_0, \xi) = \sum_{\{C(L, L_0+dL_0)\}} \sum_{\{C(L_0+dL_0, L_0)\}^*} \exp\{-[\mathcal{H}[C(L, L_0+dL_0)]$$
$$+ \mathcal{H}[C(L_0+dL_0, L_0)]]\} \tag{5.2}$$

where $\{C(L_0+dL_0, L_0)\}^*$ denotes the set of configurations of droplets of scale size $L_0 \rightarrow L_0 + dL_0$ embedded in a space of droplets of scale sizes intermediate between $L_0 + dL_0$ and L, and having some fixed configuration $C(L, L_0+dL_0)$. Denote by $V_\alpha[C(L, L_0+dL_0)]$ the volume of the space V occupied by α-coloured ($\alpha = $ B or W) droplets in the configuration $C(L, L_0+dL_0)$ of droplets with scale sizes in the range $L_0 + dL_0$ to L. Clearly

$$\sum_\alpha V_\alpha[C(L, L_0+dL_0)] = V \tag{5.3}$$

independent of the specific configuration. Now let us make the 'dilute boundary' approximation: we suppose that the white volume V_W (black volume V_B) presented to the black (white) droplets of scale size $L_0 \rightarrow L_0 + dL_0$ is such that, with a high degree of probability, any such (scale $\sim L_0$) droplet found in the presented volume will be located 'far' from the latter's boundaries. With this presupposition the problem posed by the configurational sum for the smallest (scale $\sim L_0$) droplets is then precisely that addressed in § 3. Invoking the results of that analysis (specifically (4.1) and (4.2)),

we find

$$\sum_{\{C(L_0+dL_0,L_0)\}^*} \exp\{-\mathscr{H}[C(L_0+dL_0,L_0)]\}$$

$$= 1 + V_W[C(L,L_0+dL_0)]dS_d^{-1}L_0^{-d}\psi(L_0,\xi)\,dL_0$$

$$+ V_B[C(L,L_0+dL_0)]dS_d^{-1}L_0^{-d}\psi(L_0,\xi)\,dL_0 + O(dL_0^2). \tag{5.4}$$

The second and third terms reflect, respectively, the contributions made by configurations in which one black (white) droplet, of scale size $L_0 \to L_0 + dL_0$, is found embedded in the available white (black) space presented by the larger droplets; the first term is associated with the configuration in which no such droplet is present. Notice that for (5.4) to be meaningful it is essential that the overall scale of the single-droplet partition function (i.e. the measure of the functional integral (2.12)) is correct (i.e. consistent with a zero-droplet configurational sum which is precisely unity). The quantitative reliability of the results which follow in this paper (though not their qualitative structure) thus rests squarely on the arguments substantiating the measure (2.12), as set out in appendix 1. Notice too that the temperature of the assembly appears in (5.4) only implicitly in the scale-invariant length ξ. Now, substituting (5.4) into (5.2) and recalling the sum rule (5.3), we find

$$Z(L,L_0,\xi) = \sum_{\{C(L,L_0+dL_0)\}} \exp\{-\mathscr{H}[C(L,L_0+dL_0)]\}[1 + VdS_d^{-1}L_0^{-d}\psi(L_0,\xi)\,dL_0]$$

$$= Z(L,L_0+dL_0,\xi)[1 + VdS_d^{-1}L_0^{-d}\psi(L_0,\xi)\,dL_0]$$

implying the differential equation

$$(1/Z)\partial Z(L,L_0,\xi)/\partial L_0 = -VdS_d^{-1}L_0^{-d}\psi(L_0,\xi) \tag{5.5}$$

with solution

$$Z(L,L_0,\xi) = Z(L,L,\xi)\exp\left(VdS_d^{-1}\int_{L_0}^{L}dR\,R^{-d}\psi(R,\xi)\right). \tag{5.6a}$$

The 'partition function' $Z(L,L,\xi)$ is a sum over all droplet-less configurations and is determined by the physical boundary condition. In the limit of an infinitesimal field there is but one such configuration and

$$Z(L,L,\xi) = 1 \tag{5.6b}$$

whence, recalling (4.1), we find from (5.6) the results

$$Z \equiv Z(L,L_0,\xi) = e^{Z_1}, \qquad F \equiv F(L,L_0,\xi) \equiv \ln Z = Z_1. \tag{5.7a, b}$$

Deferring discussion of the conceptual significance of these results, let us pursue the implications of (5.7b). Recalling (4.1)–(4.4) and writing $V = S_d d^{-1}L^d$, we find in the thermodynamic limit ($L/\xi \to \infty$)

$$L^{-d}F \approx \int_{L_0}^{\infty}dR\,R^{-d-1}\tilde{\psi}(R/\xi)$$

$$= \xi^{-d}\left(\int_{L_0/\xi}^{1}dz\,z^{-d-1}\tilde{\psi}(z) + \int_{1}^{\infty}dz\,z^{-d-1}\tilde{\psi}(z)\right). \tag{5.8}$$

Recalling the small-z expansion of $\tilde{\psi}(z)$, equation (4.4c), we see that

$$\int_{L_0/\xi}^{1} dz\, z^{-d-1}\tilde{\psi}(z)$$

$$= \int_{L_0/\xi}^{1} dz\, z^{-d-1}\left(\tilde{\psi}(z) - \sum_{n=0}^{n_u} \psi_n x^{n/\nu}\right) + \sum_{n=0}^{n_u} \psi_n \int_{L_0/\xi}^{1} dz\, z^{-d-1+n/\nu}. \qquad (5.9)$$

Choosing n_u such that

$$n_u > d\nu - 1$$

ensures that the first integral on the RHS of (5.9) is convergent in the critical limit ($L_0/\xi \to 0$). In this limit we then find that the free energy density (5.8) may be written in the form

$$L^{-d}F \approx f_S + f_{NS}$$

Here

$$f_S \approx f_0 \xi^{-d}, \qquad\qquad\qquad\qquad\qquad\qquad (5.10a)$$

with

$$f_0 = \left[\int_1^\infty dz\, z^{-d-1}\tilde{\psi}(z) + \int_0^1 dz\, z^{-d-1}\left(\tilde{\psi}(z) - \sum_{n=0}^{n_u} \psi_n z^{n/\nu}\right)\right], \qquad (5.10b)$$

represents the sole non-analytic contribution, while f_{NS} contains terms with (at most) analytic dependence upon the reduced temperature (parametrised by $\xi^{-1/\nu}$).

The form of the singular part of the free energy is consistent with the expectation of strong scaling theory

$$f_S \sim \xi^{-(2-\alpha)/\nu} \qquad \text{with} \qquad 2-\alpha = d\nu \qquad\qquad (5.11a, b)$$

and (cf the universality of the function $\tilde{\psi}$) bears out the universal relationship between critical-point amplitudes which is a corollary of this scaling law (Stauffer *et al* 1972, Hohenberg *et al* 1976). These are the first of many explicit expressions we shall encounter of the manifestly scaling character of our theory.

Now let us return to (5.7). These results coincide precisely with those of the 'naive' (dilute gas) droplet model. This correspondence is at first sight surprising. The 'naive' results presuppose droplets of only one phase (black, say) embedded in a background of the opposite phase, and do not account for the implied excluded volume: they thus neglect both the effects which (cf § 5.1) we identified as important in the critical region. On the other hand, the arguments leading to (5.7) *do* allow for both of these effects: they recognise that a droplet of one phase may always house subdroplets of the opposite phase (cf the second and third terms in (5.4)) and take explicit account of excluded volume effects (cf the factors of V_W and V_B in (5.4)). The reason for the coincidence of the results is that (as regards the zero-field thermodynamic properties) the two effects cancel! Specifically, each spurious (black droplet within black droplet) configuration included in the naive calculation by virtue of its neglect of excluded volume (but correctly excluded in the second calculation) makes the same contribution to the partition function as some allowed (white droplet within black droplet) configuration, erroneously omitted from the naive calculation by virtue of its neglect of droplet nesting (but correctly included in the second calculation).

Equations (5.7) are thus intelligible. Their utility is limited, however, in two respects. Firstly, their zero-field character means that they do not yield information regarding the order parameter and its susceptibility. Secondly, their form does not readily permit one to check the self-consistency of the (dilute droplet boundary) approximation on which they rest. These deficiencies may be rectified by devising a procedure which will generate explicitly the canonical ensemble of configurations underlying (5.7).

The elements of the procedure are implicit in the arguments we have already deployed in this section—in particular in (5.4) which accords to the function $\psi(R, \xi)$ an immediate physical significance: according to (5.4), $\psi(R, \xi)\,dR$ defines the mean fraction of available space occupied by droplets of scale size $R \to R + dR$. Our configuration generating procedure follows from this premise. Specifically, we take an ensemble of hyperspherical regions of radius L; these regions define our ensemble of Ising systems, so that L is to be regarded as a macroscopic length which will generally be large compared with the correlation length. We take these regions to be uniformly white; this choice of 'boundary condition' plays the role of the choice of sign for the infinitesimal field necessary to define uniquely the low-temperature phase of an Ising system. We then decorate these regions with successively smaller and smaller droplets of both black and white. At each stage of this differential dressing procedure we utilise black (white) droplets of the scale sizes $(R \to R + dR)$ then appropriate in such numbers as to yield a mean fractional coverage $\psi(R, \xi)\,dR$ of the existing white (black) space. Although the largest droplets possible in principle have scale sizes $\sim L$, the form of $\psi(R, \xi)$ (equation (4.3)) ensures that the largest possible in practice have scale sizes $\sim \xi \ll L$. The decoration procedure is terminated when the dressing droplet size reaches the arbitrary lower limit, L_0.

In the remainder of § 5 we shall set about the exploration and justification of this procedure. We shall look firstly at the degree of long-range order (the order parameter) of the configurations which the procedure generates. Armed with the results and insights thus acquired, we shall then turn to establish that the generating procedure, and the droplet picture it implies, are indeed legitimised by the smallness of ψ_0 in low space dimensions.

5.3. Application of the procedure: the order parameter

Consider an ensemble of configurations constructed by decorating an assembly of initially white L-sized regions with black and white droplets of all scale sizes $\geq L_0$, in the fashion described in § 5.2. We associate with each configuration a quantity ϕ measuring the amount by which the fraction of the decorated space occupied by white exceeds the fraction occupied by black, in the given configuration. The ensemble average $\langle \phi \rangle_{L, L_0, \xi}$ may then be taken as the order parameter characterising the degree of long-range order of the ensemble.

It is helpful to write

$$Q(L, L_0, \xi) \equiv \langle \phi \rangle_{L, L_0, \xi} = 1 - 2\Psi(L, L_0, \xi) \tag{5.12}$$

where Ψ represents the mean (ensemble average) fraction of space coloured black. To determine the manner in which Q evolves with the approach to criticality, we study the fashion in which Ψ evolves with the minimum droplet size, L_0. Specifically, we imagine dressing each configuration in our ensemble with droplets of scale sizes $L_0 \to L_0 - |dL_0|$. A fraction $\psi(L_0, \xi)|dL_0|$ of the available black space, which itself

forms a fraction $\Psi(L, L_0, \xi)$ of the whole, will then be recovered with white dressing droplets; an identical fraction of the white space, then occupying a fraction $1 - \Psi(L, L_0, \xi)$ of the whole, will be decorated with black dressing droplets. It follows that

$$d\Psi(L, L_0, \xi) = -\Psi(L, L_0, \xi)\psi(L_0, \xi)|dL_0| + [1 - \Psi(L, L_0, \xi)]\psi(L_0, \xi)|dL_0|$$

whence

$$\partial\Psi(L, L_0, \xi)/\partial L_0 = -[1 - 2\Psi(L, L_0, \xi)]\psi(L_0, \xi) \tag{5.13}$$

which has the solution

$$\Psi(L, L_0, \xi) = \tfrac{1}{2}\left[1 - \exp\left(-2\int_{L_0}^{L}\psi(R, \xi)\,dR\right)\right] \tag{5.14}$$

where we have taken the boundary condition

$$\Psi(L, L_0, \xi) = 0. \tag{5.15}$$

Now recalling the scaling form for $\psi(R, \xi)$ (equation (4.4a)), we find that in the thermodynamic limit Ψ itself assumes the scaling form

$$\Psi(L, L_0, \xi) \simeq \check{\Psi}(L_0/\xi) = \tfrac{1}{2}\left[1 - \exp\left(-2\int_{L_0/\xi}^{\infty} dz\, z^{-1}\tilde{\psi}(z)\right)\right]. \tag{5.16}$$

In the critical limit $(L_0/\xi \ll 1)$ we may write

$$\int_{L_0/\xi}^{\infty} dz\, z^{-1}\tilde{\psi}(z) = \int_{L_0/\xi}^{1} dz\, z^{-1}\psi_0 - \tfrac{1}{2}\ln Q_0 \tag{5.17a}$$

with

$$Q_0 \simeq \exp\left(-2\int_{0}^{1} dz\, z^{-1}(\tilde{\psi}(z) - \psi_0) - 2\int_{1}^{\infty} dz\, z^{-1}\tilde{\psi}(z)\right) \tag{5.17b}$$

to within corrections that are analytic in the reduced temperature. Now, combining (5.12), (5.16) and (5.17), we find for the order parameter the limiting critical behaviour

$$Q(L, L_0, \xi) = \exp\left(-2\int_{L_0/\xi}^{\infty} dz\, z^{-1}\tilde{\psi}(z)\right) = Q_0(\xi/L_0)^{-\beta/\nu} \tag{5.18a}$$

with

$$\beta/\nu \equiv 2\psi_0. \tag{5.18b}$$

Taken together, (3.28c) and (5.18b) define the two critical exponents necessary (within the strong scaling theory realised in our droplet framework) to specify the complete set of critical exponent values. Unlike ν, whose inverse appears (cf § 8) to have a power series expansion in $\varepsilon = d - 1$, the exponent β carries the essential singularity of the droplet concentration parameter ψ_0.

From a strictly numerical point of view (5.18b) is disappointing. Taking the case $d = 2$ $(\varepsilon = 1)$ and assigning to ν its appropriate exact value $\nu = 1$ (Onsager 1944), we find $\beta \simeq 0.02$ in poor accord with the exact result $\beta = 0.125$. The discrepancy is scarcely surprising in view of the sensitivity of the value of ψ_0 to the manner in which (4.4d) is extrapolated to $d = 2$. At the same time, there is some encouragement to be found in the observation that the exact $d = 2$ ratio $\beta/2\nu = 0.0625$ goes some way towards substantiating the smallness of ψ_0 and thus (cf § 5.4) the appropriateness of the droplet picture in two dimensions.

The chief value of (5.18b), however, lies at a conceptual level, which we shall explore further in § 7. For the moment we turn to examine in more detail the consistency of the framework on which the result is based.

5.4. *The consistency of the generating procedure*

In this section we shall explore more fully the configuration-generating procedure introduced and applied in preceding sections. Our specific aim is to justify the claim made earlier that the procedure is internally consistent for small enough values of the parameter ψ_0. This aim may best be realised with the aid of the droplet distribution function $n_\alpha(R, \xi)$ defined such that $n_\alpha(R, \xi)\,dR$ is the mean number per unit volume of droplets having scale size $R \to R + dR$ and colour $\alpha = W$ or B. Recalling the generating procedure, we see that the numbers $n_\alpha(R, \xi)\,dR$ will reflect the fractions of the overall space that are available at the stage where the R-sized droplets are to be utilised, together with the fractional coverage appropriate for R-sized droplets. Explicitly, taking the embedding space to be white, we find

$$n_W(R, \xi) = \Psi(L, R, \xi)v_0^{-1}(R)\psi(R, \xi), \qquad (5.19a)$$

$$n_B(R, \xi) = (1 - \Psi(L, R, \xi))v_0^{-1}(R)\psi(R, \xi), \qquad (5.19b)$$

where $v_0(R)$ (equation (4.2)) is the mean volume of droplets of scale size R (equation (4.7)) *prior* to their decoration with smaller droplets; we shall refer to $v_0(R)$ as the *bare* droplet volume. Recalling (4.4a) and (5.16), we see that in the thermodynamic limit the droplet distributions assume the scaling forms

$$n_\alpha(R, \xi) = v_0^{-1}(R)R^{-1}\tilde{n}_\alpha(R/\xi) \qquad (5.20a)$$

where

$$\tilde{n}_\alpha(z) = \tfrac{1}{2}\left[1 \pm \exp\left(-2\int_z^\infty dy\, y^{-1}\tilde\psi(y)\right)\right]\tilde\psi(z), \qquad \alpha = \frac{B}{W}. \qquad (5.20b)$$

To utilise these results one must appreciate that the actual connected volume of any given droplet remaining after subsequent decoration with smaller droplets differs from the bare value v_0. We may express this 'dressed' volume in the form

$$V(R, L_0, \xi) = v_0(R)(1 - \Psi_0(R, L_0, \xi)) \qquad (5.21)$$

where $\Psi_0(R, L_0, \xi)$ is the total fraction of a given droplet of scale size R which is covered by smaller droplets. A little thought reveals that Ψ_0 satisfies a differential equation (cf (5.13))

$$\partial\Psi_0(R, L_0, \xi)/\partial L_0 = -(1 - \Psi_0(R, L_0, \xi))\psi(L_0, \xi) \qquad (5.22)$$

whence, taking the boundary condition $\Psi_0(R, R, \xi) = 0$,

$$\Psi_0(R, L_0, \xi) = 1 - \exp\left(-\int_{L_0}^R dR'\, \psi(R', \xi)\right), \qquad (5.23)$$

$$V(R, L_0, \xi) = v_0(R)\exp\left(-\int_{L_0}^R dR'\, \psi(R', \xi)\right). \qquad (5.24)$$

Within the critical region this result assumes a scaling form

$$V(R, L_0, \xi) = v_0(R)(R/L_0)^{-\psi_0}\exp\left(-\int_0^{R/\xi} dz\, z^{-1}(\tilde{\psi}(z) - \psi_0)\right) \qquad (5.25)$$

to within correction terms that are analytic in the reduced temperature.

Finally, we shall find it useful to define a function $n_{B1}(R, \xi)$ such that $n_{B1}(R, \xi)\, dR$ is the mean number of droplets having scale size $R \rightarrow R + dR$ and colour opposite to that of the embedding space, which are not themselves housed within still larger droplets in that space (i.e. droplets which may be 'reached' from the boundaries of the embedding space without traversing any interface). In close correspondence with (5.19*b*) we find for this number density of 'one-subdroplets' the form

$$n_{B1}(R, \xi) = (1 - \Psi_0(L, R, \xi))v_0^{-1}(R)\psi(R, \xi). \qquad (5.26)$$

In § 6 we will explore several different aspects of these results; our present purpose, however, is to use them to substantiate the overall consistency of the procedure we have devised, and to expose the basic characteristics of the configurations which it generates.

To substantiate the generating procedure, we must clarify and justify the essential ('dilute droplet boundary') approximation on which it is based. To this end let us recall the elements of the generating procedure. At each state of the procedure one uses dressing droplets (of each phase) with such frequency as to cover that fraction of the available space (the space then occupied by the opposite phase) prescribed by the function $\psi(R, \xi)$ appropriate to the scale size R of the dressing droplets. The significance we have accorded to the function $\psi(R, \xi)$ is based upon its role in (5.4) which presupposes that the volume available for the dressing droplets is sufficiently large and compact to justify the neglect of boundary effects implicit in our calculation of Z_1. In fact, the volume available for dressing with droplets of scale size R will be *fragmented* in a fashion prescribed by the distribution of larger droplets already present. It is then clear that our use of the function $\psi(R, \xi)$ is justifiable only if the degree of fragmentation is not *so* great that the bulk of the available space actually consists of regions too ramified to accommodate an R-sized droplet. More precisely, the procedure will be self-consistent if and only if a dressing droplet, randomly sited within the available space, will, with a high degree of probability, have no intersections with the boundaries of that space. At first sight it would appear that this condition is bound to be violated if the generating procedure is taken to a sufficiently advanced stage, since each stage generally results in the addition of still further droplets, producing a still higher degree of fragmentation of the available space. The situation is redeemed, however, by the fact that, as the dressing programme proceeds, and the space becomes more and more highly populated with droplet boundaries, the scale size of the droplets to be accommodated is also being systematically reduced. We now establish that this trade-off is, indeed, sufficient to vindicate our procedure.

Consider a region of (initially) white space of volume V and scale size L dressed with droplets of scale sizes $\geq L_0$ with $L \gg \xi \gg L_0$. Now suppose that we extend the dressing procedure to include the droplets of scale size $L_0 \rightarrow L_0 - |dL_0|$. To legitimise our procedure it is sufficient to examine the dressing of the white space (then available) with black droplets; the dressing of the black space with white droplets may be treated in a similar fashion. It will be helpful to address the problem in two stages. Consider first the portion of the white space which is connected to the system boundary (i.e.

the 'background' space which is not covered by 'one-subdroplets'). The mean volume of this connected white space follows from (5.21) and (5.23) as

$$V_{CW} = V(1 - \Psi_0(L, L_0, \xi)) \sim V(L_0/\xi)^{\psi_0} \tag{5.27}$$

which actually vanishes in the critical limit: one-subdroplets actually fill the entire system space in the asymptotic critical limit. (This result is less disconcerting when one recalls that each such one-subdroplet itself has droplet substructure.) Now a black dressing droplet, of scale size L_0, implanted in this connected white space will intersect an existing boundary if its centre lies within a distance $\sim L_0$ of such a boundary. The portion of the connected white space which is dangerous in this regard (i.e. the portion which lies within L_0 of an existing droplet surface) has volume

$$\Delta V_{CW} = L_0 V \int_{L_0}^{L} dR \, n_{B1}(R, \xi) a(R, L_0, \xi) \tag{5.28}$$

where $a(R, L_0, \xi)$ is the mean surface area of droplets of scale size R, considered in § 4.3. (We have reparametrised the results of this section to eliminate T in favour of ξ.) Invoking (4.17) and (5.26) we find

$$\Delta V_{CW} \sim \psi_0 V(L_0/\xi)^{\psi_0}. \tag{5.29}$$

Evidently, then the ratio $\Delta V_{CW}/V_{CW}$ is $O(\psi_0)$. It follows that, as the dressing procedure evolves, the use of the concentration function ψ remains legitimate in all but a vanishingly small (ψ_0-sized) fraction of the connected white space.

This argument is readily extended to legitimise the use of the function ψ to characterise the dressing in the entire white (or black) space. Specifically one finds for the total volume of white space in the limit $L \gg \xi \gg L_0$

$$V_W = V(1 - \Psi(L, L_0, \xi)) \simeq V/2 \tag{5.30}$$

reflecting the fact that, in this critical limit, the space is equally populated with black and white. The portion of this (fragmented!) white space which lies within a distance $\sim L_0$ of an existing droplet surface has volume

$$\Delta V_W = L_0 V \int_{L_0}^{L} dR \, (n_W(R, \xi) + n_B(R, \xi)) a(R, \xi, L_0) \sim \psi_0 V \tag{5.31}$$

again indicating that our use of the function ψ will result only in an error of order ψ_0.

These arguments are, we believe, sufficient to justify the generating procedure we have devised. However, there are a number of complementary observations which may usefully be made at this point, since they will serve to illuminate how the smallness of ψ_0 may be invoked to resolve the many apparent paradoxes one discovers in utilising the generating scheme.

Specifically, let us now enquire in what sense it can be meaningful to dress an existing droplet with subdroplets (of the opposite phase) which are only infinitesimally smaller. The answer lies in the fact that, in the configurations yielded by our procedure, the only subdroplets which will be found in our bounding droplet in any abundance will actually have scale sizes very small compared with that of the bounding droplet. To be explicit, consider a ('bounding') droplet of scale size R_0 embedded in a space which is dressed with droplets of all scale sizes $\geq L_0 = fR_0$, where $f < 1$. Let us suppose that $R_0 \ll \xi$, so that the population of droplets embedded within the bounding droplet is at its maximum, and our problem is presented in its most acute form. We will first

of all establish how small f has to be in order that the mean number of one-subdroplets to be found in the bounding droplet should be equal to unity. Recalling (5.26), we see that f is prescribed by the requirement

$$1 = v_0(R_0) \int_{fR_0}^{R_0} dR \, n_{B1}(R, \xi)$$

$$= \int_{fR_0}^{R_0} dR \, (1 - \Psi_0(R_0, R, \xi))(R_0/R)^d \psi(R, \xi)$$

$$\approx (\psi_0/d) f^{-d + \psi_0}$$

whence

$$f \approx (\psi_0/d)^{1/(d - \psi_0)}. \tag{5.32}$$

Secondly, let us enquire how small f has to be to ensure that a (mean) fraction g of the bounding droplet is covered by one-subdroplets. Evidently

$$g = \Psi_0(R_0, fR_0, \xi)$$

from which we find that

$$f \approx (1 - g)^{1/\psi_0}. \tag{5.33}$$

These two results vindicate the answer we have already preferred to the question posed above, and clarify the character of the critical-point droplet configurations. The picture that emerges is this. In the critical limit our configurations are built from statistically self-similar droplets of the two phases. Each droplet has within it a 'lowerarchy' of embedded droplets whose scale sizes are bounded below by the cut-off length L_0. Although (to the extent that its scale size is large compared with L_0) each droplet is essentially covered with its one-subdroplets, the largest of these is typically very small compared with the given droplet's scale size, and the extent to which the boundaries of these one-subdroplets intersect one another (or intersect the boundary of the given droplet) is small. In both instances the term 'small' has a precise significance set by the value of ψ_0.

6. Droplet number distributions

In this section we shall explore the droplet number distributions determined in § 5. We shall see that both the forms and properties of these distributions differ in a number of illuminating respects from those anticipated on the basis of droplet phenomenologies.

We begin by observing that the distribution functions satisfy the two consistency requirements

$$1 = 1 - \Psi_0(L, L_0, \xi) + \int_{L_0}^{L} dR \, n_W(R, \xi) V(R, L_0, \xi) + \int_{L_0}^{L} dR \, n_B(R, \xi) V(R, L_0, \xi), \tag{6.1a}$$

$$Q(L, L_0, \xi) = 1 - \Psi_0(L, L_0, \xi) + \int_{L_0}^{L} dR \, n_W(R, \xi) V(R, L_0, \xi)$$

$$- \int_{L_0}^{L} dR \, n_B(R, \xi) V(R, L_0, \xi). \tag{6.1b}$$

The first term on the RHS of each equation represents the fraction of the original white space remaining after dressing with droplets of scale sizes $\geqslant L_0$. The second term represents the fraction of space occupied by white droplets. The third term gives (in magnitude) the fraction of space occupied by black droplets. The first equation merely checks 'space conservation'; the second displays consistency with our earlier results (5.12), (5.14) for the order parameter.

In demonstrating that $(6.1a, b)$ are, indeed, satisfied with the identifications prescribed in (5.19), (5.23) and (5.24) one discovers that interesting cancellations occur. Specifically, a number of terms on the RHS of $(6.1b)$ which vanish critically as $|t|^{\beta/2}$ (given the identification (5.18)) cancel to recover the original result. These cancellations are at the heart of a significant contradiction between the explicit forms of the distribution functions, obtained here, and the forms postulated on the basis of phenomenological arguments. To expose this contradiction let us consider the key equation of the phenomenological framework, the droplet representation of the order parameter. When suitably recast, the phenomenological arguments yield an equation with the following structure (Binder 1976):

$$Q(L, L_0, \xi) = 1 - 2 \int_{L_0}^{\infty} dR\, R^{-\lambda-1} \tilde{q}(R/\xi) \qquad (6.2)$$

with $\tilde{q}$ an unknown scaling function, and with the exponent λ prescribed by

$$\lambda = \beta/\nu. \qquad (6.3a)$$

With the aid of (5.20), (5.23), (5.24) and $(6.1a)$ one finds that the result $(6.1b)$ can indeed be recast in the form (6.2), in the thermodynamic limit. The scaling function $\tilde{q}$ is then explicitly prescribed as

$$\tilde{q}(z) = L_0^{\psi_0} \tilde{n}_B(z) \exp\left(-\int_0^z dy\, y^{-1}(\tilde{\psi}(y) - \psi_0)\right).$$

However, in contrast to $(6.3a)$ one finds that the exponent λ appearing in (6.2) must be identified as

$$\lambda = \psi_0 = \beta/2\nu \qquad (6.3b)$$

where we have made use of $(5.18b)$.

To identify the origins of this contradiction let us review the argument by which, within the phenomenological framework, one infers the result $(6.3a)$, given the general form (6.2). The argument requires that one match the anticipated power law behaviour of the order parameter to that predicted by (6.2). This matching is most conveniently effected by differentiating (6.2) with respect to ξ:

$$\partial Q(L, L_0, \xi)/\partial \xi = 2\xi^{-2} \int_{L_0}^{\infty} dR\, R^{-\lambda} \tilde{q}'(R/\xi) = 2\xi^{-\lambda-1} \int_{L_0/\xi}^{\infty} dz\, z^{-\lambda} \tilde{q}'(z). \qquad (6.4)$$

Comparison with the anticipated behaviour

$$\partial Q(L, L_0, \xi)/\partial \xi \sim \xi^{-\beta/\nu-1} \qquad (6.5)$$

does indeed seem to suggest the identification $(6.3a)$, with the limiting $(L_0/\xi \to 0)$ value of the integral in (6.4) defining the amplitude of the power law. However, this reasoning presupposes that the integral in (6.4) remains finite in the critical limit; in fact, at least within the droplet theory explicitly realised here, the value of the integral

vanishes in the critical limit. Specifically one finds that, with the explicit form for $\tilde{q}$ prescribed above, and with the identification (6.3b),

$$\int_{L_0/\xi}^{\infty} dz \, z^{-\lambda} \tilde{q}'(z) \simeq -\psi_0 Q_0 L_0^{2\psi_0} \xi^{-\psi_0} \tag{6.6}$$

where Q_0 is the amplitude appearing in (5.18a). Combining this result with (6.4), one finds immediately the same prediction as that yielded by differentiating the power law (5.18a) directly, confirming the consistency of the identification (6.3b).

We have no clear understanding of the physical significance of the sum rule on $\tilde{q}'$ which dictates that the integral (6.6) should vanish in the critical limit, although it is tempting to associate it with the more obvious sum rule which $\tilde{q}$ must satisfy in order that the order parameter (equation (6.2)) should itself vanish at criticality. Nevertheless, the moral at least is clear: one cannot reliably identify the exponents characterising the scaling properties of phenomenological droplet distribution functions by matching to expressions which are sensitive only to the distribution function moments.

Thus far we have shown that the identification (6.3a) is not mandatory and that the identification (6.3b) is internally consistent. We now proceed to show that the latter identification, which emerges naturally within our explicit theory, actually serves to resolve a problem characteristic of the phenomenological identification. Specifically, we consider the second moment of the droplet size distribution

$$s = \int_{L_0}^{\infty} dR \, n_B(R, \xi) V^2(R, L_0, \xi) \tag{6.7}$$

whose singular behaviour is that of the mean droplet size. Recalling (5.19b) and (5.25), we find that s may be written in the form

$$s = \int_{L_0}^{\infty} dR \, R^{-2\lambda+d-1} \tilde{s}(R/\xi) \tag{6.8}$$

where $\tilde{s}$ is a prescribed function. Again the structure of this result coincides with that of the simple phenomenological droplet theory; again, however, our explicit theory requires that λ be prescribed by (6.3b) rather than (6.3a). The consequences for the singular behaviour of the mean droplet size are significant. According to (6.8)

$$s \approx s_0 \xi^{\theta/\nu} \qquad \text{where } \theta = (d-2\lambda)\nu \tag{6.9a, b}$$

is the 'droplet size exponent'. Invoking the relation

$$\gamma = d\nu - 2\beta \tag{6.10}$$

which is necessarily fulfilled within any strong scaling theory (cf § 7), we find that while the droplet phenomenology (6.3a) implies

$$\theta = \gamma \tag{6.11a}$$

our explicit theory yields

$$\theta = \gamma + \beta. \tag{6.11b}$$

Now, in setting up our theory we argued that Ising configurations may be built from nested droplets, in low dimensions where the smallness of ψ_0 ensures that the troublesome boundary-overlap effects are small. In exploring the implications of (6.11a, b) we must enquire to what extent our putative droplet building blocks can

actually be identified in a given configuration, since the identification is a clear prerequisite for a 'measurement' of the droplet size and its exponent. We have already noted in § 5.1 that, in $d = 3$, this identification is problematic in view of the percolation of islands (more precisely 'nearest-neighbour clusters') of black spins through a white ordered phase, at a temperature below T_c: the mean cluster size thus diverges at this percolation point, and not at the critical point, in three dimensions (Müller-Krumbhaar and Stoll 1976, Sykes and Gaunt 1976). In the $d = 2$ nearest-neighbour Ising model, however, the coincidence of the (zero-field) percolation point and the critical point is mandated by the topology (Coniglio *et al* 1977), and the mean cluster size does diverge critically, engendering the expectation that the critical properties of 'clusters' might directly correspond with those of 'droplets'. Hitherto this identification has foundered on the fact that, according to series expansion studies of the triangular lattice (Sykes and Gaunt 1976), the mean cluster size diverges with an exponent $\theta = 1.91 \pm 0.01$ which is irreconcilable with the phenomenological prediction $(6.11a)$, $\theta = \gamma = 1.75$. We see, however, that the observed value *is* close to the prediction $(6.11b)$ of our explicit droplet theory, $\theta = \gamma + \beta = 1.875$. (We have made use of exact $d = 2$ exponent values.) The small remaining discrepancy may be no more than an indication that it is difficult to obtain long enough series to do full justice to the nested droplet structure. In this context it is pertinent to note that the mean bare cluster size (i.e. the mean volume of 'one-subdroplets', cf (5.26)) has the critical behaviour of the quantity

$$s' = \int_{L_0}^{\infty} dR \, n_{B1}(R, \xi) v_0^2(R) \sim \xi^{\theta'/\nu} \qquad (6.12a)$$

with

$$\theta' = d\nu = \gamma + 2\beta \qquad (6.12b)$$

which has the value $\theta' = 2$ in $d = 2$. It follows then, that a study which, by accident or design, includes in the 'cluster size' the entire space contained within the cluster boundaries (space which, it will now be appreciated actually accommodates still further clusters) should yield a cluster size exponent higher than that predicted by $(6.11b)$. This observation is consistent with the sign of the discrepancy noted above; further studies are clearly necessary to establish whether the discrepancy should indeed be understood in this way.

In summary it seems reasonable to interpret the success of the prediction $(6.11b)$ as a favourable reflection on both our droplet model itself and the supplementary assumption (which is not intrinsic to the framework of our theory) that our droplet properties are directly mirrored in the properties of nearest-neighbour Ising model clusters in $d = 2$. This interpretation is, however, preferred with reserve in view of the fact that there exist at least two alternative theories of the observed cluster size behaviour whose relation to our own is not fully clear.

Firstly, we observe that the prediction $(6.11b)$ coincides with a conjecture by Stauffer (1977) based on the 'generalised' droplet phenomenology developed by Binder (1976). It would appear that this phenomenology may be regarded as an attempt to allow for the existence of droplet substructure; in this respect it is in keeping with the spirit of the present theory. However, although this phenomenology does embrace the prediction $(6.11b)$ (as one special case), its predictions for the exponents characterising the scaling form of the number distribution function n_B remain inconsistent with our own for the reasons set out earlier in this section.

Secondly, we note that Coniglio and Klein (1980) have suggested a strategy by which one may reconcile the observed behaviour of lattice Ising model configurations with the predictions of the simple droplet theory (specifically, (6.11a)). They propose a prescription by which the clusters of the lattice Ising model may be broken into smaller clusters, whose percolation point (for any dimension) does coincide with the Ising model critical point. They present a RG argument indicating that the mean size of these clusters diverges in the manner prescribed by the simple droplet result (6.11a). We have not been able to see how this interesting result should be viewed within the present framework.

7. Ordering coordinate fluctuations

Thus far we have restricted our attention to properties (typified by the order parameter) which are sensitive only to the mean numbers of droplets of each scale size appearing in the Ising ensemble. We now extend our study to include properties (typified by the order parameter susceptibility) which, to varying degrees, reflect the fluctuations in droplet numbers about their mean values.

The concept of 'number fluctuations' requires some elaboration. Specifically, we must recall that the function $\psi(R, \xi)$ around which our configuration-generating procedure is constructed, prescribes only the mean fraction of available space to be occupied by droplets of scale sizes $R \to R + dR$, and thus the mean number of droplets with scale sizes in that range. Accordingly the droplet distributions which we have established are to be thought of as describing the relevant droplet numbers averaged over an ensemble of configurations generated by our procedure. In any given member of the ensemble the number of droplets of a given scale size will not, in general, coincide with its mean value. It is the nature and implication of such fluctuations which we shall now explore.

Let us define the specific problem to be addressed. Consider the black droplets with scale sizes in the range $R_s \to R_s + dR_s$ found embedded in an ensemble of Ising samples each of which is initially white, has a macroscopic volume V, and is subsequently dressed with droplets of all scale sizes $\geq L_0$, where $L_0 < R_s$. We associate with each black R_s-sized droplet a 'droplet coordinate' ϕ_{R_s} giving the amount by which the fraction of the droplet space that is white exceeds the fraction that is black. Our aim is to characterise the distribution of ϕ_{R_s} values.

The motivation is twofold. Firstly, we will find that the moments of the distribution furnish additional predictions for critical exponents (specifically, the susceptibility exponent γ). Secondly we shall see that the probability density function (PDF) for the coordinate ϕ_{R_s}, which we shall calculate in the critical limit $\xi \gg R_s \gg L_0$, yields further insight into the nature of the critical-point configurations.

The droplet coordinate PDF may be studied most conveniently with the aid of the associated characteristic function

$$\hat{p}(H; R_s, L_0, \xi) \equiv \langle \exp(iH\phi_{R_s}) \rangle_{L_0, \xi} \tag{7.1}$$

whose logarithm has an expansion in the cumulants of the ϕ_{R_s} distribution (see e.g. Kendall and Stuart 1963)

$$\ln \hat{p}(H; R_s, L_0, \xi) = \sum_{n=1}^{\infty} \frac{(iH)^n}{n!} \Phi^{(n)}(R_s, L_0, \xi) \tag{7.2a}$$

with

$$\Phi^{(n)}(R_s, L_0, \xi) = \langle \phi_{R_s}^n \rangle_{L_0, \xi}^{(c)} \tag{7.2b}$$

where the superscript (c) denotes connected part. Adopting the strategy employed in § 5.3, we will determine the characteristic function (7.1) by setting up and solving the differential equation which describes the way in which the function evolves with the minimum dressing size L_0.

Consider, then, a particular R_s-sized droplet characterised by a particular value of the coordinate ϕ_{R_s}. Now imagine dressing the ensemble (and thus this droplet in particular) with further droplets of scale size $L_0 \to L_0 - |dL_0|$. Let the numbers of white (black) droplets utilised to decorate the given droplet in this infinitesimal process be $N_W(N_B)$. The change in the value of the coordinate ϕ_{R_s} (for the selected droplet) is then

$$d\phi_{R_s} = 2(N_W - N_B)(L_0/R_s)^d \tag{7.3}$$

where we have made use of the form of the bare droplet volume, (4.2). A little thought now reveals that the characteristic function for the droplet coordinate ϕ_{R_s} in the new ensemble thus generated may be written in the form

$$\hat{p}(H; R_s, L_0 - |dL_0|, \xi) = \langle\!\langle \exp\{iH[\phi_{R_s} + 2(N_W - N_B)(L_0/R_s)^d]\} \rangle\!\rangle \tag{7.4}$$

where the notation is supposed to indicate an averaging procedure in which one
   (i) averages over the spectrum of N_W and N_B values appropriate for a given ϕ_{R_s} and
   (ii) averages over all values of ϕ_{R_s} in the original ensemble.

Consider the first of these averaging procedures. The rationale here is that our ensemble will contain many R_s-sized droplets characterised by the same specific value of ϕ_{R_s}. Our generating procedure dictates that the mean numbers of black and white droplets to be utilised in the dressing of R_s-sized droplets characterised by a given ϕ_{R_s} should be

$$\langle N_W \rangle = \tfrac{1}{2}\psi(L_0, \xi)(1 - \phi_{R_s})(R_s/L_0)^d |dL_0|, \tag{7.5a}$$

$$\langle N_B \rangle = \tfrac{1}{2}\psi(L_0, \xi)(1 + \phi_{R_s})(R_s/L_0)^d |dL_0|. \tag{7.5b}$$

Now in the differential limit the probability that any given R_s-sized droplet (with coordinate ϕ_{R_s}) will receive zero droplets of species α is $1 - \langle N_\alpha \rangle$; the probability that it receives precisely one droplet of species α is $\langle N_\alpha \rangle$; the probability that it receives more than one droplet is negligible. The first averaging procedure then yields

$$\langle\!\langle \exp\{iH[\phi_{R_s} + 2(N_W - N_B)(L_0/R_s)^d]\}\rangle\!\rangle$$
$$= \langle \exp(iH\phi_{R_s})\{1 + |dL_0|\psi(L_0, \xi)(R_s/L_0)^d [\cos(2H(L_0/R_s)^d) - 1$$
$$- i\phi_{R_s}\sin(2H(L_0/R_s)^d)]\}\rangle.$$

Recalling the defining equation (7.1) for the characteristic function, one may now readily perform the remaining average over ϕ_{R_s}. The results may be expressed in terms of the differential equation

$$\partial \ln \hat{p}(H; R_s, L_0, \xi)/\partial L_0$$
$$= -\psi(L_0, \xi)(R_s/L_0)^d \{\cos(2H(L_0/R_s)^d)$$
$$- 1 - \sin(2H(L_0/R_s)^d)(\partial/\partial H) \ln \hat{p}(H; R_s, L_0, \xi)\}. \tag{7.6}$$

The boundary conditions for this equation are dictated by our dressing procedure

which ensures that, for $L_0 = R_s$, the coordinate $\phi_{R_s} = -1$ for all (black) R_s-sized droplets in the ensemble, so that

$$\hat{p}(H; R_s, R_s, \xi) = e^{-iH}. \tag{7.7}$$

Equations (7.6) and (7.7) constitute the central results of this section. To explore their implications we make use of the cumulant expansion for the characteristic function, (7.2a). The boundary condition (7.7) then assumes the form

$$\Phi^{(n)}(R_s, R_s, \xi) = -\delta_{n,1} \tag{7.8}$$

while (7.6) yields an infinite series of coupled differential equations satisfied by the cumulants.

Let us first consider the equation satisfied by the cumulant $\Phi^{(1)}$ which follows on equating terms linear in H in the cumulant representation of (7.6)

$$\partial\Phi^{(1)}(R_s, L_0, \xi)/\partial L_0 = -2\psi(L_0, \xi)\Phi^{(1)}(R_s, L_0, \xi). \tag{7.9}$$

With the boundary condition (7.8) this equation has the solution

$$\Phi^{(1)}(R_s, L_0, \xi) \equiv \langle\phi_{R_s}\rangle_{L_0,\xi} = -\exp\left(-2\int_{L_0}^{R_s} dR\, \psi(R, \xi)\right)$$

$$= -[1 - 2\Psi(R_s, L_0, \xi)]. \tag{7.10}$$

For $R_s \gg \xi$, equation (7.10) shows that $\langle\phi_{R_s}\rangle$ is independent of R_s and (cf (5.12)) coincides with the order parameter for a system characterised by parameters ξ, L_0 (and having 'black' boundary conditions). This result expresses the fact that any droplet with scale size large compared with ξ may itself be regarded as an Ising system. On the other hand, for $\xi \gg R_s, L_0$ we find from (7.10), (5.14)

$$\Phi^{(1)}(R_s, L_0, \xi) \approx -\exp\left(-2\psi_0\int_{L_0}^{R_s} dR\, R^{-1}\right) = -(L_0/R_s)^{2\psi_0} \tag{7.11}$$

a result which we shall utilise in due course.

The differential equation satisfied by the second cumulant is obtained by equating the $O(H^2)$ terms in the expansion of (7.6):

$$\partial\Phi^{(2)}(R_s, L_0, \xi)/\partial L_0 = 4\psi(L_0, \xi)[\Phi^{(2)}(R_s, L_0, \xi) - (L_0/R_s)^d] \tag{7.12}$$

The general solution is

$$\Phi^{(2)}(R_s, L_0, \xi) = \exp\left(-4\int_{L_0}^{R_s} dR\, \psi(R, \xi)\right)$$

$$\times \left[\text{Constant} + 4R_s^{-d}\int_{L_0}^{R_s} dR'\, \psi(R', \xi)(R')^d \exp\left(4\int_{R'}^{R_s} dR\, \psi(R, \xi)\right)\right]. \tag{7.13}$$

Invoking the boundary condition (7.8), we find

$$\Phi^{(2)}(R_s, L_0, \xi) = 4R_s^{-d}\int_{L_0}^{R_s} dR'\, \psi(R', \xi)(R')^d \exp\left(-4\int_{L_0}^{R'} dR\, \psi(R, \xi)\right). \tag{7.14}$$

Let us examine this result first of all in the limit $R_s \gg \xi, L_0$. In this limit we find

$$\Phi^{(2)}(R_s, L_0, \xi) \approx \Phi_0^{(2)}(\xi/L_0)^{d-4\psi_0} \tag{7.15}$$

with

$$\Phi_0^{(2)} = 4(R_s/L_0)^{-d} \int_0^\infty dz\, \tilde{\psi}(z) z^{d-1-4\psi_0} \exp\left(-4 \int_0^z dy\, y^{-1}[\tilde{\psi}(y) - \psi_0]\right). \tag{7.16}$$

Now, as we have already noted, in the prescribed limit, an R_s-sized droplet itself represents an Ising system (in the thermodynamic limit). According to the central limit theorem (see e.g. Baker and Krinsky 1976) the coordinate ϕ_{R_s} must then be normally distributed, with a variance (second cumulant) which is vanishingly small on the scale of the squared mean (to the extent that $R_s \gg \xi$) and which is a direct measure of the susceptibility. Specifically we anticipate

$$\Phi^{(2)}(R_s, L_0, \xi) \approx v(R_s)^{-1}\chi. \tag{7.17}$$

Comparison of (7.16) and (7.17) yields the identifications

$$\chi \approx \chi_0(\xi/L_0)^{\gamma/\nu} \tag{7.18a}$$

with

$$\gamma = (d - 4\psi_0)\nu, \qquad \chi_0 = v_0(R_s)\Phi_0^{(2)}. \tag{7.18b, c}$$

Equation (7.18b) bears out the scaling relation anticipated in (6.10).

Now let us return to examine (7.14) in the limit $\xi \gg R_s$. In this region we find

$$\Phi^{(2)}(R_s, L_0, \xi) \approx \frac{4\psi_0}{d - 4\psi_0}\left(\frac{L_0}{R_s}\right)^{4\psi_0}\left[1 - \left(\frac{L_0}{R_s}\right)^{d-4\psi_0}\right].$$

Imposing the additional condition $R_s \gg L_0$, we obtain

$$\Phi^{(2)}(R_s, L_0, \xi) \approx (4\psi_0/d)(L_0/R_s)^{4\psi_0} \tag{7.19}$$

where, in keeping with the philosophy of our calculation, we have neglected $O(\psi_0^2)$ contributions to the prefactor. Comparison of (7.11) and (7.19) reveals that, in the limit $\xi \gg R_s \gg L_0$,

$$\Phi^{(2)}(R_s, L_0, \xi)/[\Phi^{(1)}(R_s, L_0, \xi)]^2 \simeq 4\psi_0/d. \tag{7.20}$$

We defer discussion of this result and proceed, instead, to substantiate the expectation which it engenders, namely that in the regime $\xi \gg R_s \gg L_0$ the droplet coordinate distribution assumes a non-trivial universal form.

Consider, then, the differential equations generated by equating the terms of order H^n in the cumulant expansion of (7.6). For n even we find, in the prescribed regime,

$$\frac{\partial \Phi^{(n)}(R_s, L_0, \xi)}{\partial L_0} = 2n\frac{\psi_0}{L_0}\Phi^{(n)}(R_s, L_0, \xi) - 2^n\left(\frac{L_0}{R_s}\right)^{(n-1)d}\frac{\psi_0}{L_0}. \tag{7.21a}$$

We have neglected contributions to the RHS of this equation that mix in the cumulants $\Phi^{(n-2)}, \Phi^{(n-4)} \ldots$. One may readily establish that these terms merely contribute $O(\psi_0^2)$ corrections to the amplitude of the power law governing the asymptotic $(L_0/R_s \ll 1)$ behaviour of $\Phi^{(n)}$. For n odd we find within a similar approximation

$$\frac{\partial \Phi^{(n)}(R_s, L_0, \xi)}{\partial L_0} = 2n\frac{\psi_0}{L_0}\Phi^{(n)}(R_s, L_0, \xi) + 2^n\left(\frac{L_0}{R_s}\right)^{(n-1)d}\frac{\psi_0}{L_0}\Phi^{(1)}(R_s, L_0, \xi) \tag{7.21b}$$

where we have retained the contribution made by $\Phi^{(1)}$ since it, alone amongst the cumulants, exhibits power law behaviour (as function of L_0/R_s) with an amplitude

that is $O(1)$ (rather than $O(\psi_0)$). Equations (7.21a, b) have a structure similar to that of (7.12) and may be solved in a similar fashion to yield, in the regime $\xi \gg R_s \gg L_0$,

$$\Phi^{(n)}(R_s, L_0, \xi) \approx [2^n \psi_0/(n-1)d](L_0/R_s)^{2n\psi_0}, \qquad n > 1. \tag{7.22}$$

The form of the characteristic function in this regime now follows from (7.2):

$$\hat{p}(H; R_s, L_0, \xi) \approx \exp\left[-iH\left(\frac{L_0}{R_s}\right)^{2\psi_0} + \frac{\psi_0}{d} \sum_{n=2}^{\infty} \frac{(2iH)^n}{n!(n-1)}\left(\frac{L_0}{R_s}\right)^{2n\psi_0}\right]. \tag{7.23}$$

The PDF of the droplet coordinate ϕ_{R_s} follows as the Fourier transform of the characteristic function

$$P(\phi_{R_s}; R_s, L_0, \xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} dH \exp(-iH\phi_{R_s})\hat{p}(H; R_s, L_0, \xi).$$

In the prescribed regime it is clear from (7.23) that the PDF assumes a universal form when normalised to unit mean. Explicitly

$$p(\phi_{R_s}; R_s, L_0, \xi) \, d\phi_{R_s} \approx p^*(u) \, du \tag{7.24a}$$

where

$$u = (R_s/L_0)^{2\psi_0} \phi_{R_s}, \tag{7.24b}$$

$$p^*(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dh \, \exp\left(-ih(1+u) + \frac{\psi_0}{d} \sum_{n=2}^{\infty} \frac{(2ih)^n}{n!(n-1)}\right). \tag{7.24c}$$

One may show analytically that the characteristic function (7.23) vanishes exponentially for large H; accordingly the associated PDF is not singular and may be computed by numerical evaluation of the Fourier transform (7.24c). The result is shown in figure 3. We have set $d = 2$ and $\psi_0 = \beta/2\nu = \frac{1}{16}$, in keeping with (5.18$b$) and the exact values of the exponents β and ν in two dimensions. The form of the distribution prompts a number of remarks.

Firstly we note the narrowness of the distribution on the scale of its mean, already manifested analytically in the $O(\psi_0)$ value of the ratio prescribed in (7.20). The
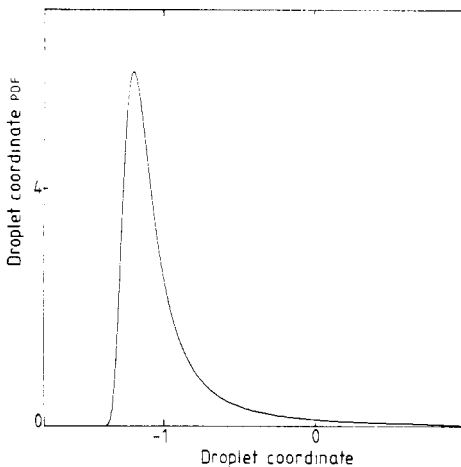

Figure 3. Probability density for a (black) droplet coordinate in the regime $\xi \gg R_s \gg L_0$, as prescribed in (7.24c) with $d = 2$, $\psi_0 = \frac{1}{16}$.

implication for the critical-point configurations is clear: droplets whose scale sizes are small compared with ξ are ordered in a fashion which is more homogeneous the lower the space dimension.

Secondly, figure 3 reveals a strong asymmetry in the manner in which the droplet coordinate is distributed about its mean value: fluctuations whose sense is opposite to that of the prevailing intra-droplet order are much more strongly favoured than fluctuations tending to enhance that order.

Finally we may usefully make contact with independent calculations based on approximation schemes quite different to that employed here, which corroborate the structure suggested in figure 3. The studies in question (Bruce 1981b, Binder 1981) are concerned with the distribution of 'block' coordinates u_{L_s} in Ising systems of various space dimensions. A block coordinate u_{L_s} gives a linear measure of the instantaneous aggregate of the local coordinates lying within a sampling volume of linear dimension L_s. For block sizes L_s small compared with ξ but large compared with all microscopic length scales, the block PDF is believed to assume a limiting form P^* which is universal when expressed on the scale set by its standard deviation (Patashinskii 1968, Bruce 1981b). The form of the function appears to depend only weakly upon the shape of the sampling volume, but strongly upon the space dimension. The nature of the function for 'low' values of $\varepsilon = d - 1$ may be determined by a straightforward extension of the arguments developed above. Specifically, we observe that the characteristic function associated with a block coordinate u_{L_s},

$$\hat{p}(H; L_s, L_0, \xi) \equiv \langle \exp(iHu_{L_s}) \rangle, \tag{7.25}$$

will satisfy a differential equation whose form is identical to that obeyed by the characteristic function associated with a droplet coordinate ϕ_{R_s} (equation (7.6)), with the substitution $R_s \rightarrow L_s$. The boundary conditions for the differential equation are, however, different from those appropriate for (7.6) and, indeed, are less immediately prescribed by the rules of our generating procedure. For the present purpose, however, the following argument will suffice. A sampling volume defined by a sphere of radius L_s, embedded in an Ising system which contains only droplets of scale sizes larger than L_s, will predominantly be entirely within a single phase region, since (cf the arguments developed in § 5.4 to justify our generating procedure) the probability of overlap with the surfaces of larger droplets is $O(\psi_0)$. Accordingly we may impose on the characteristic function (7.25) the boundary condition

$$\hat{P}(H; L_s, L_s, \xi) = \tfrac{1}{2}(e^{iH} + e^{-iH}) \tag{7.26}$$

with an error (in the cumulants of the boundary form of the PDF) that is $O(\psi_0)$. We note that the boundary condition presupposes that $\xi \gg L_s$ so that the sampling volume is equally likely to be embedded in either of the two phases. Within the approximation implicit in (7.26), one then finds that the block coordinate PDF (in the regime $\xi \gg L_s \gg L_0$) is simply a symmetrised version of the droplet coordinate PDF (7.24c)

$$P^*(u_{L_s}; L_s, L_0, \xi)\, du_{L_s} = \tfrac{1}{2}[p^*(u) + p^*(-u)]\, du \tag{7.27a}$$

with

$$u \equiv (L_s/L_0)^{2\psi_0} u_{L_s}. \tag{7.27b}$$

In view of the approximation inherent in the boundary condition (7.26) this calculation is not systematically correct to $O(\psi_0)$. Nevertheless, the general structure of the function P^* implied by this calculation for $d = 2$, $\psi_0 = \tfrac{1}{16}$ (cf figure 4) is in accord with
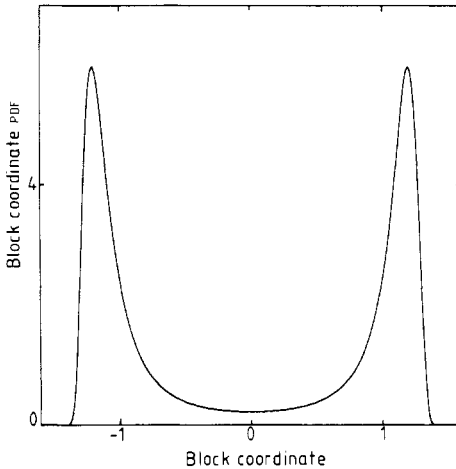
Figure 4. Probability density for a block coordinate in the regime $\xi \gg L_s \gg L_0$, as prescribed in (7.27a) with $d = 2$, $\psi_0 = \frac{1}{16}$.

independent studies of the PDF in two dimensions, based on Wilson's approximate recursion formula (Bruce 1981b) and on Monte Carlo techniques (Binder 1981): like figure 4, both the latter calculations yield a function P^* displaying two distinct markedly asymmetric peaks.

Though quantitatively less reliable than the earlier calculations, the present study offers two compensations. Firstly, it prescribes an explicit interpolation between the double-peaked structure of the function in $d = 2$ and the double δ-function form appropriate in $d = 1$; the structure of the former may thus be ascribed to nested droplets, just as the structure of the latter is controlled by kinks (Bruce 1981b). Secondly, we see that the high degree of short-range order manifest on length scales small compared with ξ, which is expressed in the narrowness of the two peaks in the PDF, is a corollary of the smallness of the exponent β characteristic of low dimensions, since both the peak width (7.20) and the exponent value (5.18b) are controlled by the single fundamental parameter ψ_0.

8. Conclusions and prospects

In this section we attempt to highlight the key insights which emerge from the work we have presented, and identify some of the questions which it leaves unresolved.

Our essential aim has been to extend the theory of the planar interface initiated by Wallace and Zia (1979) to yield a theory of phase transitions in low-dimensional Ising systems. The theory we have developed is not the first of this kind. Its most noteworthy predecessor is the bond-shifting form of the renormalisation group, introduced by Kadanoff (1976b) which, like our own, is legitimised by a small $\varepsilon = d - 1$ approximation. The work presented here complements its predecessors in a number of important respects of which the most significant is the explicit way in which it illuminates the nature of the dominant configurations and reveals their signatures in critical-point observables. The three key insights relevant in this regard may be summarised in the following way.

Firstly, we have established that the coarse-grained configurations of a low-dimensional Ising system are those of an assembly of droplets, whose boundaries remain dilute, for small enough ε, even in the critical region.

Secondly, we have seen that, in the limit in which the droplet scale size R is small compared with ξ, droplet-surface fluctuations result in a droplet surface area which has a fractal character (Mandelbrot 1977), as appreciated by David (1981). The fractal dimension, d_a, of the surface is linked to the correlation length exponent ν (cf (4.17)):

$$a(R, L_0, \xi) \sim R^{d_a}, \qquad d_a = 1/\nu. \qquad (8.1a, b)$$

Thirdly, we have established that, as a result of droplet-nesting, the dressed droplet volume also has a fractal character, in the limit $R \ll \xi$. The fractal dimension of the droplet volume is linked to the order parameter exponent β (cf (5.25)):

$$V(R, L_0, \xi) \sim R^{d_V}, \qquad d_V = d - \beta/2\nu. \qquad (8.2a, b)$$

Returning to the first of these points, we re-emphasise that it is the diluteness of the droplet boundaries, guaranteed by small values of the concentration parameter ψ_0, that seems to us to be the distinctive simplifying feature of low-dimensional configurations. We have already noted (and show more explicitly in appendix 2) that this feature may be regarded as a natural corollary of the diluteness of cluster walls (kinks) in one dimension. We have seen that it legitimises an explicit and tractable realisation of the nested droplet picture, which in turn suggests the existence of crucial flaws in related phenomenologies (cf (6.3a, b)).

Turning now to the remaining two points made above we note that, although the use of the term 'fractal' in the Ising-model droplet literature has been sparse, the appreciation of the essential content of (8.1) and (8.2), at a qualitative level at least, certainly predates the coining of the term. The earliest droplet theories (Fisher 1967) allow for the possibility that a cluster (of reversed spins, or of molecules in the liquid-gas context) might have a surface area, a, scaling with its volume V,

$$a \sim V^{\sigma_a} \qquad (8.3)$$

in a non-Euclidean fashion (i.e. $\sigma_a \neq 1 - 1/d$). The idea of a 'fractal' droplet volume is also clearly implicit in the later variants of the droplet picture (Stauffer *et al* 1971, Kadanoff 1976a) which recognise the essential role of droplet nesting.

These ideas have figured rather more explicitly, however, in the closely related context of the percolation problem, where a number of differently defined fractal dimensions have been introduced to characterise clusters near the percolation threshold. The profusion of dimensionality assignments has been surveyed by Stauffer (1979); we will limit our remarks to a comparison of the prediction (8.2) with the assignment of the percolation cluster fractal dimension favoured by a majority of authors. Specifically, if one defines such a dimension $d_{V,p}$ by requiring that the mean fraction of space occupied by (mean fraction of sites belonging to) the infinite cluster, for $p \gtrsim p_c$, in a region of linear dimension $R \ll \xi$, should scale as $R^{d_{V,p}}$ one finds immediately that $d_{V,p}$ is related to the percolation exponents β_p and ν_p by

$$d_{V,p} = d - \beta_p/\nu_p. \qquad (8.4)$$

Since the 'infinite cluster' in the percolation problem is analogous to the dressed volume of a single large droplet, one might expect a closer correspondence between the fractal dimensions d_V and $d_{V,p}$ than is suggested by (8.2b) and (8.4). In fact, the significance of the difference between the two forms is less than meets the eye since

the exponents β and β_p play different roles in the two problems. In the percolation problem the order parameter exponent (β_p) is defined such that the mean fraction of space occupied by (number of sites belonging to) the percolating cluster vanishes as $\xi^{-\beta_p/\nu_p}$. In the Ising problem, on the other hand, the order parameter exponent (β) is defined such that the difference between the fraction of space occupied by one phase and the fraction ocupied by the other vanishes as $\xi^{-\beta/\nu}$. If, however, in the spirit of the percolation problem, one defines an Ising model order parameter exponent β' such that the mean fraction of space occupied by the 'infinite dressed droplet' (which forms the background sea of the dominant phase) vanishes as $\xi^{-\beta'/\nu}$, one finds that

$$\beta' = \beta/2 \tag{8.5a}$$

so that the fractal dimension $(8.2b)$ may equally well be written as

$$d_V = d - \beta'/\nu. \tag{8.5b}$$

While the formal correspondence of (8.4) and $(8.5b)$ may seem reassuring, it is largely empty: each relationship is little more (given a scaling theory, *no* more) than a definition of the corresponding order parameter exponent. On the other hand, $(8.5a)$ (which may be viewed as a corollary of $(6.11b)$) does have some non-trivial content: the dressed volume of a droplet, and the difference between the total volumes occupied by the two phases of which it consists, scale with exponents which differ by a factor of 2. Indeed, Monte Carlo studies of the infinite cluster in the ordered phase of a $d = 2$ nearest-neighbour Ising model suggest critical behaviour consistent with $(8.5a)$ (Jan *et al* 1982).

While the geometrical interpretation of the two principal critical exponents β and ν afforded by (8.1) and (8.2) is certainly appealing, it clearly can hold only in a limited range of space dimensions. In particular, the constraint $d_a \geq d - 1$ is not fulfilled in $d = 3$, and is fulfilled only as an equality in $d = 2$ if one makes the assignment $(8.1b)$. Indeed, it is not hard to identify the possible sources of these limitations.

We observe, firstly, that, although the arguments which lead through (4.17) to the result (8.1) can formally be extended order by order in ε, since these arguments ignore the renormalising effects of other droplets it is not at all clear how they can be justified except in the dilute regime guaranteed by the smallness of ψ_0 at low $d - 1$. In particular one might expect that, in addition to its intrinsic structure (generated by surface fluctuations), a droplet surface will be embellished with further detail associated with other droplets, with which it has a partial overlap. Equation $(8.1b)$ should thus be written as

$$d_a = 1/\nu + O(\psi_0) \tag{8.6a}$$

where we may expect the $O(\psi_0)$ corrections to be positive.

Secondly, as regards the limitations of $(8.2b)$ we remark that our neglect of the renormalising effects of surface fluctuations (implicit in the spherical approximation for the bare droplet volume) seems not to be a problem since these fluctuations do not appear to change the volume scaling dimension (cf (4.7)). Thus the only approximation inherent in $(8.2b)$ is again the presupposition of dilute boundaries so that, more precisely, we assert

$$d_V = d - (\beta/2\nu)(1 + O(\psi_0)). \tag{8.6b}$$

To the extent that the properties of droplets are reflected in the behaviour of lattice Ising model clusters (cf the extensive discussion in § 6), the two predictions (8.6a) and (8.6b) are susceptible to test by Monte Carlo simulations—most immediately through their implication for the surface exponent σ_a defined in (8.3). Recalling (8.1a) and (8.2a) we see that

$$\sigma_a = d_a/d_V = (d\nu - \beta/2)^{-1} + O(\psi_0) \tag{8.7}$$

where, strictly, the β term should be discarded since it is of the same order as the neglected terms $O(\psi_0)$. Equation (8.7) yields $\sigma_a = \frac{16}{31}$ for $d = 2$. This result is actually consistent with that reported by Binder and Stauffer (1972) whose Monte Carlo studies indicated a value of σ_a close to $\frac{1}{2}$. However, the latter study was restricted to temperatures $T \leq 0.9T_c$ and its relevance to the 'asymptotic' regime must be viewed with reserve. Indeed, a later Monte Carlo study (Domb et al 1975) probing a region closer to criticality ($T \approx 0.96T_c$) reported a value of $\sigma = 0.75 \pm 0.05$. However, in this case it appears that the surface area of a cluster was taken to include that associated with internal boundaries. The different scaling properties of this total surface area A are actually implicit in (4.17) and (5.28) which show that (again for $R \ll \xi$)

$$A(R, L_0, \xi) \sim R^{d_A}, \qquad d_A = d - (\beta/2\nu)(1 + O(\psi_0)). \tag{8.8a, b}$$

The correspondence between (8.6b) and (8.8b) then implies that

$$\sigma_A \equiv d_A/d_V = 1. \tag{8.9}$$

in accord with a suggestion by Stauffer (1975). That the value of σ reported by Domb et al (1975) is larger than that recorded by Binder and Stauffer (1972) is thus intelligible. That the value falls short of the prediction (8.9) is presumably a reflection of the fact that the latter can be expected to hold only in the regime of small enough reduced temperatures and large enough cluster sizes that the contributions of internal surfaces are dominant. Indeed, Domb et al (1975) themselves observe that their σ assignment should be regarded as an effective rather than asymptotic exponent value, and record a trend towards larger σ values for larger cluster sizes. Clearly, further Monte Carlo studies are now in order to check the extent to which the characteristics of droplets implied by results such as (6.11b) and (8.7) are borne out in the properties of lattice Ising model clusters in two dimensions.

This brings us, appropriately, to the point where we must identify some of the issues which this work has left unresolved, and consider the prospects for further study.

Firstly, it will certainly be of some interest to establish the extent to which the calculations reported here can be extended to higher order. It would appear that the prospects for the refinement of the two parts, into which this work falls naturally, are rather different. It seems clear that the theory of the nearly spherical isolated droplet can be extended systematically, just as in the case of the nearly planar isolated interface (Wallace and Zia 1979, Forster and Gabrinas 1981). It is less clear that our multi-droplet theory can be extended in any systematic fashion to include the effects of the intersection of droplet boundaries. This limitation is possibly more serious than it might seem since, as remarked earlier, it is far from clear that calculations of 'isolated' droplet properties—specifically, the determination of the exponent ν from single-droplet-surface characteristics—can be trusted outside the dilute regime. We

expect that the situation will be similar to that in one dimension, where (Zinn-Justin 1981) the value of asymptotic series must be defined before exponentially small effects become significant except for physical quantities which have zero asymptotic expansion. Thus we anticipate that the class of exponents (ν, α) will also acquire exponentially small contributions from droplet interactions (overlap) whose value can be given only when the sum of the asymptotic series in ε is defined. The leading exponentially small effects in the class $\beta, \eta \ldots$ are significant because these exponents have zero asymptotic expansion. In fact, the existence of effects of this kind seems an inescapable corollary of the fact that the isolated droplet theory predicts an exponent ν for the q-state Potts model that is independent of q (Schmittmann 1982). However, an explicit resolution of this issue seems likely to prove difficult, requiring, as it will, a greater degree of unification of the two portions of this work than we have achieved here.

In this context we should now recall that while a form of lattice spacing (minimum droplet size) appears naturally in our theory of multi-droplet configurations, our single-droplet theory, being dimensionally regularised, lacks this reassuring feature. A reworking of the latter calculation on a lattice might be informative, but would be complicated by the fact that an explicit lattice cut-off breaks important symmetries of the Hamiltonian (2.11).

There are, however, other extensions of our theory which may prove more tractable, and are certainly of interest.

First, it is clearly desirable to establish the extent to which the theory of coexistence curve ($H = 0$, $T < T_c$) behaviour presented here can be extended to other regions of the H–T plane. In the region $T < T_c$, $H \neq 0$ we foresee no difficulties of principle: a theory of this regime will allow one to unify the treatment of the critical region given here with the theory of the first-order transition on the subcritical portion of the coexistence curve developed elsewhere. The regime $T > T_c$, however, is more problematic. Below the critical temperature the effects of droplets of all scale sizes are controlled by virtue of the fact that $T(R)$ is bounded between the low-temperature and critical fixed points, 0 and $T_c = O(\varepsilon)$. Above the critical temperature, $T(R)$ is not bounded for large R and it is not clear to us how one may characterise the effects of large droplets.

Secondly, it seems likely that the arguments presented here may also be extended to embrace the percolation problem. Indeed, Gefen *et al* (1981) have recently proposed a scheme for generating a fractal representation of the backbone of the percolating cluster, which (scheme) shares with our droplet generating scheme the flavour of a RG transformation run in a reverse mode. Furthermore, Schmittmann (1982) has already considered the generalisations of our arguments necessary to describe the q-state Potts model, whose $q = 1$ limit is now well known to generate the physics of the percolation problem (see e.g. Stauffer 1979).

Thirdly, we have some hopes that the present theory may be extended to yield a more explicit theory of correlations than we have developed here. In particular, it remains to be demonstrated that at large distances the correlation function does indeed have the anticipated exponential decay over a length which (for suitable choice of the amplitude c_0) coincides with the 'correlation length' ξ prescribed in (3.28b).

Finally, with the encouragement of a recent study of time-dependent properties of the isolated planar interface (Bausch *et al* 1981) we may also view optimistically the prospects for a dynamical droplet theory with which one may explore the strongly nonlinear dynamics of low-dimensional Ising systems.

Appendix 1. Functional measure of the single droplet partition function

The calculation of the single-droplet partition function Z_1 in § 3 is tied to the use of dimensional regularisation in two important aspects.

First, since the expression (2.12) for Z_1 involves the absolute value of a single functional integral, the result apparently depends on the convention chosen for the scale of the field $f(\eta)$ or its spherical harmonic amplitudes $a_{l\alpha}$. In particular, if we had chosen a measure $\Pi_{l,\alpha}(\zeta\, da_{l\alpha})$ in (3.5) where ζ in some arbitrary constant, the value obtained for Z_1 would change by a factor ζ^y where $y = \Sigma_l \nu_l(d)$. However, for all values of d for which this sum converges one can show from (2.9) that its value is zero. Hence in dimensional regularisation $y = 0$ and the result is independent of the scale of $f(\eta)$ in the measure.

Second, it is implicit in the calculation in § 3 that the measure Df is invariant under the transformation (2.8) on the field f, corresponding to a rigid translation of the droplet. In general, one might expect that a non-trivial Jacobian factor would be needed to achieve this invariance. If we write the infinitesimal translation (2.8) in the form $f' = f + F(f)$, the condition $DfJ(f) = D(f')J(f')$ for an invariant measure becomes $J(f) = \det((\delta f'/\delta f)J(f')$. Expanding this equation for infinitesimal F yields

$$-\mathrm{Tr}\,\frac{\delta F}{\delta f} \equiv -\int d\Omega\, \delta(\eta - \eta')\frac{\delta F(f(\eta))}{\delta f(\eta')} = \int d\Omega\, \frac{\delta \ln J}{\delta f(\eta)}F(f(\eta)). \qquad (A1.1)$$

From (2.8), for translation by infinitesimal amount x,

$$F(f(\eta)) = x_i\eta_i + x_i\eta_j L_{ij} \ln(R + f(\eta))$$

and hence

$$\delta F(f(\eta))/\delta f(\eta')) = x_i\eta_j L_{ij}[[R + f(\eta)]^{-1}\delta(\eta - \eta')].$$

Upon integration by parts the reader may then check that the functional

$$J = C \exp\frac{d-1}{d}\delta(0)\int d\Omega \ln(R + f)$$

is formally a solution of (A1.1), where C is an arbitrary constant. Thus indeed a Jacobian factor is required to ensure an invariant measure $DfJ(f)$. However, this Jacobian factor contains $\delta(0)$ which is identically zero in the framework of dimensional regularisation so that in that scheme J can be taken as a constant.

In the calculation of § 3, however, we explicitly assumed that the constant C was 1. In large measure this assumption was initially based on intuitive notions of what the sum over all surfaces means in e.g. lattice models. At this point this is a very unsatisfactory state of affairs because although the functional form of Z_1 would be independent of C its universal amplitude and hence the exponent β would depend on C.

In order to place the result (3.30) for Z_1 on a sounder footing, we consider the calculation in the framework of the Landau–Ginzburg model of a one-component field $\phi(x)$:

$$H = \int d^d x [\tfrac{1}{2}(\nabla\phi)^2 + V(\phi)] \tag{A1.2}$$

where some cut-off representing a lattice or mean molecular spacing is to be understood. Since we are concerned here with describing two competing phases below a critical temperature, $V(\phi)$ should have two competing minima at ϕ_+ and ϕ_- say, associated with the two possible phases. We can attempt to make the droplet configurations in (A1.2) explicit by writing the formal expansion for the partition function

$$Z = Z_0\left(1 + \sum_{n=1}^{\infty} Z_n\right) \tag{A1.3}$$

where

$$Z_0 = \int_0 D\phi \, e^{-H} \tag{A1.4a}$$

$$Z_n = Z_0^{-1} \int_n D\phi \, e^{-H}, \qquad n \geq 1, \tag{A1.4b}$$

and the label n indicates that the associated functional integral extends over all configurations containing n droplets. For a general potential $V(\phi)$ (with two competing minima at $\phi_\pm$) there is of course considerable ambiguity in the decomposition (A1.3). For example, when does a 'fluctuation' from a homogeneous phase qualify as a 'droplet'? This ambiguity is removed in the 'deep well' or 'thin wall' limit. In this limit the potential $V(\phi)$ generates a δ-function (Ising-like) distribution for ϕ; in the simple case where

$$V(\phi) = -\tfrac{1}{2}\mu^2\phi^2 + \tfrac{1}{4}g\phi^4 - H\phi \tag{A1.5}$$

this corresponds to $\mu \to \infty$, $g \to \infty$ with g/μ^2 fixed. Then the only feature left is the statistical mechanics of the sharp interfaces separating regions with ϕ_+ and ϕ_-.

In principle, all the calculations of this paper could be carried out within the framework of this model, with the parameter μ acting as a regulator for the short-distance fluctuations in the surfaces. We would expect that for an arbitrary potential $V(\phi)$ (with external field $H = 0$) the one-droplet partition function Z_1 in (A1.4b) would in the limit $\mu \to \infty$ give the universal result (3.30). The technicalities in performing such a calculation for Z_1 are substantial (see e.g. Gervais and Neveu (1976)) and we have not attempted to do so.

However, to provide some justification for this claimed universality, we have followed Langer (1967) and set up a calculation of the imaginary part of Z_1 obtained from (A1.4b) in the metastable region, in $(1+\varepsilon)$ dimensions. The heart of this calculation involves expanding the field ϕ in the numerator $\int_1 D\phi$ about the critical droplet solution ϕ_c and in the denominator $\int_0 D\phi$ about the metastable homogeneous extremum ϕ_0 of H. Neglecting anharmonic terms in the expansions, one has formally

$$Z_1 = \exp -[H(\phi_c) - H(\phi_0)] \, \mathrm{Det}^{-1/2}[(-\nabla^2 + V''(\phi_c))/(-\nabla^2 + V''(\phi_0))] \tag{A1.6}$$

The exponential factor contains the classical droplet energy and the ratio of determinants contains the one-loop corrections. For the droplet theory in this paper to be

physically sensible, (A1.6) must yield a universal expression equivalent to (3.30) in the deep well limit of any potential V. This limit has been considered in some detail by Affleck (1979). Noting that in this limit the solution ϕ for a droplet of radius R_c approaches a step function

$$\phi_c \simeq \phi_- + (\phi_+ - \phi_-)\theta(r - R_c),$$

we have calculated the ratio of determinants for the special case

$$V''(\phi_c) = \mu_-^2 + (\mu_+^2 - \mu_-^2)\theta(r - R_c) + \lambda\delta(r - R_c), \qquad (A1.7a)$$

$$V''(\phi_0) = \mu_+^2 \qquad (A1.7b)$$

Here the curvatures (mass terms) in the stable and metastable wells of V at ϕ_- and ϕ_+ are denoted by μ_-^2 and μ_+^2; they are taken to infinity in the deep well limit. The δ-function potential has a coefficient λ whose value is given in terms of μ_+^2 and μ_-^2 by the constraint that the translation mode $\partial_\mu\phi_c$ must be an exact $l = 1$ eigenfunction of $-\nabla + V''(\phi_c)$ with eigenvalue zero.

We have calculated (A1.6) for the special case (A1.7) in the limit $\mu_+, \mu_- \to \infty$, with λ fixed by the above constraint. The resulting expression could in principle depend on λ; we find that the imaginary part of Z_1 is independent of λ and has the same value as would be obtained from the claimed result (3.20) by analytic continuation into the metastable region. Thus, within the class of potentials (A1.7), we substantiate the normalisation for the functional integral (2.12). The extension of the argument to a general potential $V(\phi)$ is conceivable but has not been attempted here.

Appendix 2. The one-dimensional limit

In this appendix we consider the special limiting case of one dimension. Our aim is to show that the surface fluctuation model developed in this paper reproduces, in this limit, the results predicted by the domain wall phenomenologies known to capture the basic physics of the Ising universality class in $d = 1$ (Krumhansl and Schrieffer 1975, Aubry 1975, Currie *et al* 1980).

Consider first of all the correlation length ξ. Inserting into (3.27) the $\varepsilon = 0$ form of the β function (3.26a), one finds as the $d = 1$ analogue of (3.28b)

$$\xi = c_1 L_0(T(L_0))^{1/2} \exp[1/T(L_0)]. \qquad (A2.1)$$

The exponential term in this expression reflects the influence of the classical droplet surface energy, and is controlled by the coefficient of the $O(\varepsilon)$ contribution to ν^{-1} (equation (3.28c)). The prefactor originates in the effects of droplet surface fluctuations, and is controlled by the coefficient of the $O(\varepsilon^2)$ contribution to ν^{-1}. The amplitude c_1 is a constant which sets the overall length scale and, like the constant c_0 in (3.28a) (to which it is related by a factor which is a singular function of ε), is not prescribed by our theory.

Remarkably, the form of (A2.1) is in precise accord with the results of studies of ϕ^4 and related models in one dimension (Currie *et al* 1980), which yield the same functional dependence of the correlation length ξ upon the temperature T (expressed in units of the domain wall energy) as is prescribed by (A2.1), modulo system-specific (non-universal) prefactors playing the role of our own unspecified amplitude c_1. One cautionary comment is, however, in order here. The temperature-dependent ($T^{1/2}$)

prefactor predicted in (A2.1) is not a universal feature of the $d = 1$ Ising universality class: although it appears in the correlation length for Ising systems in the continuum limit in which the kink width is large compared with the lattice spacing, it is not present in the fixed length spin Ising limit. We do not fully understand why the $d = 1$ limit of our theory realises the former situation rather than the latter.

Now let us examine the $d = 1$ limit of the function $\psi(R, \xi)$. Substituting into (4.3) the solution for $T(R)$ found by integrating the $d = 1$ form of the β-function equation (3.26a), we find

$$\psi(R, \xi) = c_2(R/L_0^2)(T(L_0))^{-1} \exp[-2/T(L_0)][1 + O(T(R))] \qquad (A2.2)$$

where c_2 is a dimensionless non-universal constant, prescribed by the renormalisation scheme. Now, according to the arguments advanced in § 5.1, equation (A2.2) (more precisely, $\psi(R, \xi) \, dR$) should give the mean fraction of space occupied by 'one-dimensional' droplets of scale size $R \to R + dR$. A one-dimensional droplet of scale size R is simply a kink–antikink pair, separated by an interval R which contains no other domain walls. Recalling that the probability per unit length of finding a domain wall is $1/2\xi$, one sees that the probability of finding a droplet of scale size $R \to R + dR$ is $dR/4\xi^2$, per unit length, provided that $R \ll \xi$ so that the constraint that there be no intervening domain walls can be neglected. The mean number of such droplets in length L is then $L \, dR/4\xi^2$, and the mean fraction of space which they occupy will thus be $(R/L)(L \, dR/4\xi^2)$. We conclude that

$$\psi(R, \xi) = R/4\xi^2, \qquad R \ll \xi. \qquad (A2.3)$$

Recalling (A2.1), it is clear that the surface fluctuation model prediction (A2.2) is consistent with the prediction of the domain wall theory (A2.3), and that the two results may be brought into full accord by assigning an appropriate value to the unknown amplitude c_1. The failure of (A2.3) for $R \gtrsim \xi$ is also implicitly corroborated by (A2.2). In the region above the critical point (the region to which we have 'continued' our theory in the course of this section—necessarily so, since $t_c = 0$ in $d = 1$) the coupling constant $T(R)$, and thus the $O(T(R))$ 'corrections' to (A2.2), are unboundedly large for large R (cf § 8).

References

Affleck I K 1979 *PhD Thesis, Harvard University*
Amit D J 1978 *Field theory, the Renormalisation Group and Critical Phenomena* (New York: McGraw-Hill)
Andreev A F 1964 *Sov. Phys.-JETP* **18** 1415
Aubry S 1975 *J. Chem. Phys.* **62** 3217
Baker G A Jr and Kim D 1980 *J. Phys. A: Math. Gen.* **13** L103
Baker G A Jr and Krinsky S 1976 *J. Math. Phys.* **18** 590
Bausch R, Dohm V, Janssen H K and Zia R K P 1981 *Phys. Rev. Lett.* **47** 1837
Bervillier C 1976 *Phys. Rev.* B **14** 4964
Binder K 1976 *Ann. Phys., NY* **98** 390
—— 1981 *Z. Phys.* B **43** 119
Binder K and Stauffer D 1972 *J. Stat. Phys.* **6** 49
Brézin E, Le Guillou J C and Zinn-Justin J 1973 *Phys. Rev.* D **8** 434
—— 1976a *Phys. Rev.* D **14** 2615
—— 1976b *Phys. Rev.* B **14** 4976
Brézin E and Zinn-Justin J 1976a *Phys. Rev. Lett.* **36** 691
—— 1976b *Phys. Rev.: Math. Gen.* B **14** 3110

Bruce A D 1981a *Ferroelectrics* **35** 43
—— 1981b *J. Phys. C: Solid State Phys.* **14** 3667
Bruce A D and Wallace D J 1981 *Phys. Rev. Lett.* **47** 743
Coniglio A and Klein W 1980 *J. Phys. A: Math. Gen.* A **13** 2775
Coniglio A, Nappi C R, Perrugi F and Russo L 1977 *J. Phys. A: Math. Gen.* A **10** 205
Currie J F, Krumhansl J A, Bishop A R and Trullinger S E 1980 *Phys. Rev.* B **22** 477
David F 1981 *Phys. Lett.* B **102** 193
Domb C 1976 *J. Phys. A: Math. Gen.* **9** 283
Domb C, Schneider T and Stoll E 1975 *J. Phys. A: Math. Gen.* **8** L90
Drummond I T and Shore G M 1979 *Ann. Phys., NY* **121** 204
Enting I G and Baxter R J 1980 *J. Phys. A: Math. Gen.* **13** 3723
Fisher M E 1967 *Physics* **3** 255
——1974 *Rev. Mod. Phys.* **46** 597
Fisk S and Widom B 1969 *J. Chem. Phys.* **50** 3219
Forster D and Gabriunas A 1981 *Phys. Rev.* A **24** 598
Gefen Y, Aharony A, Mandelbrot B B and Kirkpatrick S 1981 *Phys. Rev. Lett.* **47** 1771
Gervais J L and Neveu A 1976 *Phys. Rep.* **23C** 237
Gradshetyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)
Günther N J, Nicole D A and Wallace D J 1980 *J. Phys. A: Math. Gen.* **13** 1755
Hohenberg P C, Aharony A, Halperin B I and Siggia E D 1976 *Phys. Rev.* B **13** 2986
Jacucci G, Perini A and Martin G 1983 *J. Phys. A: Math. Gen.* **16** 369
Jan N, Coniglio A and Stauffer D 1982 *J. Phys. A: Math. Gen.* **15** L699
Kadanoff L P 1971 in *Critical Phenomena* ed M S Green (New York: Academic) p 100
—— 1976a in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green (New York: Academic)
    vol 5a, p 1
—— 1976b *Ann. Phys., NY* **100** 359
Kendall M G and Stuart A 1963 *The Advanced Theory of Statistics* (London: Griffin and Co) vol 1
Krumhansl J A and Schrieffer J R 1975 *Phys. Rev.* B **11** 3535
Langer J S 1967 *Ann. Phys., NY* **41** 108
Lowe M J and Wallace D J 1980 *J. Phys. A: Math. Gen.* **13** L381
McKane A J 1979 *Nucl. Phys.* B **152** 166
McKane A J and Wallace D J 1978 *J. Phys. A: Math. Gen.* **11** 2285
Mandelbrot B B 1977 *Fractals; Form, Chance and Dimension* (San Francisco: Freeman)
Müller-Krumbhaar H and Stoll E 1976 *J. Chem. Phys.* **65** 4294
Onsager L 1944 *Phys. Rev.* **65** 117
Patashinskii A Z 1968 *Sov. Phys.–JETP* **26** 1126
Schmittmann B 1982 *J. Phys. A: Math. Gen.* **15** 3571
Stauffer D 1975 *J. Phys. C: Solid State Phys.* **8** L172
—— 1977 *J. Phys. A: Math. Gen.* **10** L71
—— 1979 *Phys. Rep.* **54** C1
Stauffer D, Ferer M and Wortis M 1972 *Phys. Rev. Lett.* **29** 345
Stauffer D, Kiang C S and Walker G H 1971 *J. Stat. Phys.* **3** 323
Sykes M F and Gaunt D S 1976 *J. Phys. A: Math. Gen.* **9** 2131
Wallace D J and Zia R K P 1979 *Phys. Rev. Lett.* **43** 808
Wilson K G and Kogut J 1974 *Phys. Rep.* **12C** 77
Zinn-Justin J 1981 *Nucl. Phys.* B **192** 125